The Mathematics Teacher

APRIL 1961

Interval graphing
SHELDON T. RIO *and* WALTER J. SANDERS

The Advanced Placement Program in Mathematics
RICHARD S. PIETERS *and* E. P. VANCE

HOBLADIC
GEORGE W. HORTON

Digital computer programming in high school classes
LOWELL T. VAN TASSEL

Computer programming for high schools and junior colleges
DARREL G. LITTLEFIELD

Digital computer—nonelectronic
FERD W. MCELWAIN

XUM

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XUM

Interval graphing

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Interval graphing provides fresh opportunities at all levels of secondary mathematics.

Introduction

THERE IS NO DOUBT THAT the concept of a graph and the processes of graphing comprise one of the more important subjects in the school mathematics syllabus. This importance stems from the fact that all of the properties of a relation—and therefore of a function—are incorporated in its graph, with the added feature that certain information is more easily and quickly obtainable from the graph than from the analytic expression of the relation.

Too often the unit on graphing involves only the study of special techniques for the graphing of a few standard forms, such as linear functions, circles, and perhaps a few quadratic equations, with the result that the student frequently cannot graph cases that do not fit one of these special forms.[1] Such a student becomes quite adept at plotting points when the co-ordinates are given but is incapable of determining the co-ordinates that satisfy the conditions of a nonstandard problem. In fact, the student may not even be aware that he should be looking for the "solution set" of the problem, that is, the set of co-ordinates of the points he is to graph.

The solution to this problem is not merely to devise clever methods to handle the standard forms—which the student already has the ability to graph—but is rather to devise and use more generalized exercises that illustrate the basic concepts of graphing. One set of exercises which in an interesting way not only illustrates these basic concepts of graphing, but also unifies many topics of mathematics, is based on "interval graphing."

[1] It is encouraging to note that in some cases the order relations and the absolute value and greatest integer functions are studied. (See UICSM Unit 4.)

Intervals

A closed interval, $[a, b]$, is the set of all real numbers x, such that $a \leq x \leq b$. An open interval, $]a, b[$, is the set of all real numbers x, such that $a < x < b$. It is to be noted that if b is less than a, then $[a, b]$ and $]a, b[$ contain no points. The one-dimensional intervals are not too interesting to graph since in all but the degenerate cases their graphs are line segments (with or without the endpoints as the interval is closed or open, respectively). The special case $[a, a]$ graphs as a single point while the graph of $]a, a[$ is void.

The notion of an interval can be generalized to two dimensions as an ordered pair of one-dimensional intervals. Thus $([a, b], [c, d])$ is the set of all two-dimensional points (x, y), such that $a \leq x \leq b$ and $c \leq y \leq d$. In general, for $a < b$ and $c < d$, $([a, b], [c, d])$ is a rectangle together with its interior. A similar interpretation can be given for $(]a, b[,]c, d[)$, etc. All two-dimensional intervals will be called rectangles even if they are only segments or are void.

One might suspect that this sort of building block would only lead to graphs of rectangles, but we shall see that more exotic forms, such as the interior of a circle, can be described completely with

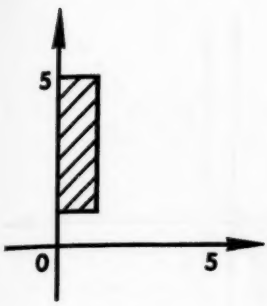


Figure 1

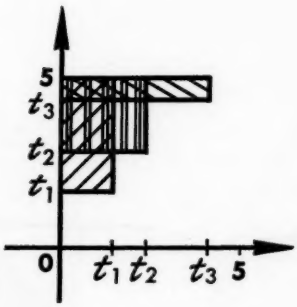


Figure 2

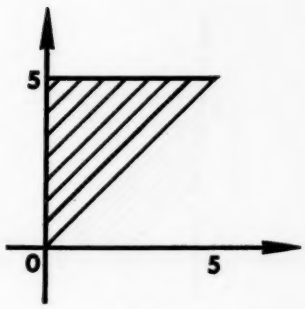


Figure 3

these two-dimensional intervals. The following examples illustrate how the graphs of more complex forms can be obtained.

EXAMPLES

Example 1. Consider the two-dimensional interval

$$([0, t],[t, 5]),$$

where t is an arbitrary real number. For $t=1$, we have the interval $([0, 1], [1, 5])$, the graph of which is shown in Figure 1. For each real number t, where $0<t<5$, the graph is a rectangle. The rectangles for three such values of t are illustrated in Figure 2. If $t=0$, the graph is merely the portion of the y-axis from 0 to 5; if $t=5$, it is the portion of the line $y=5$ from $x=0$ to $x=5$. For $t<0$ or $t>5$, the interval is void. The graph for the complete set of intervals

$([0, t], [t, 5])$, for each real number t,

is a triangle together with its interior as illustrated in Figure 3. (The reader may convince himself that the graph is the suggested triangle by $[a]$ trying to find a point within or on the triangle that is not in some interval and $[b]$ trying to find a point outside the triangle that is within some interval.)

Example 2. Graph the complete set of intervals

$$([\tfrac{1}{2}t, t], [1-\tfrac{1}{2}t, 2-t]),$$

for each real number t.

In order to determine the complete graph, a first step is to graph a few particular intervals. Some intervals are listed with their corresponding values of t in Table 1. The

TABLE 1

t	$([t/2,t], [1-t/2,2-t])$
-1	void
0	([0, 0], [1, 2])
$\frac{1}{2}$	($[\frac{1}{4}, \frac{1}{2}]$, $[\frac{3}{4}, \frac{3}{2}]$)
1	($[\frac{1}{2}, 1]$, $[\frac{1}{2}, 1]$)
$1\frac{1}{2}$	($[\frac{3}{4}, \frac{3}{2}]$, $[\frac{1}{4}, \frac{1}{2}]$)
2	([1, 2], [0, 0])
3	void

graphs of these rectangles appear in Figure 4. The position of the intervals graphed in Figure 4 suggests that the graph of the complete set of intervals is the region shown in Figure 5. Plotting intervals corresponding to additional values of t will quickly convince one that

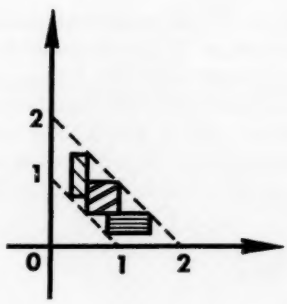


Figure 4

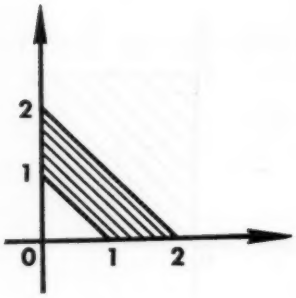


Figure 5

Figure 5 does indeed picture the graph of the set of intervals required in Example 2.

So far, the examples graphed have been bounded regions. The next example shows that the graph may extend without limit.

Example 3. The set of intervals given by

$$([t-1, t], [t^3, t]), \text{ for each real number } t,$$

is graphed in Figure 6. It is noted that for values of t greater than -1 and less than 0, or greater than 1, the intervals are void. The right-hand boundary of the curve is given parametrically by

$$\begin{aligned} x &= t \\ y &= t^3 \end{aligned} \qquad t \leq -1 \quad \text{or} \quad 0 \leq t \leq 1.$$

By eliminating the parameter t, the more familiar form

$$y = x^3, \qquad x \leq -1 \quad \text{or} \quad 0 \leq x \leq 1$$

is obtained. What are the other boundary curves?

Example 4. It was suggested previously that the interior of a circle could be generated by a set of intervals. To accomplish this, we are guided by the equation of the boundary of such a graph. For convenience, consider a circle with center at the origin and with radius 1. The equation is

$$x^2 + y^2 = 1.$$

Solving for y,

$$y = \pm\sqrt{1-x^2}.$$

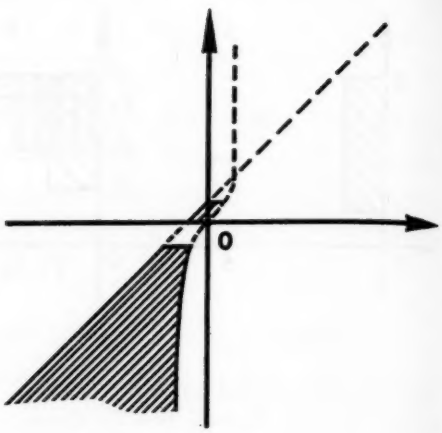


Figure 6

Parametric equations are then suggested to be

$$\begin{aligned} x &= t \\ y &= \sqrt{1-t^2} \quad \text{or} \quad -\sqrt{1-t^2}. \end{aligned}$$

The problem now is to use these parametric equations to determine expressions for the endpoints of the intervals. One suggestion is the set of intervals

$$([0, t], [0, \sqrt{1-t^2}]), \text{ for each real } t.$$

The graph of this set of intervals is only one quarter of a circular region (including the boundary). The other quarters of the circle may be obtained by including both positive and negative algebraic signs; the boundary may be omitted from the graph by using open intervals. The set of intervals,

$$(]-t, t[,]-\sqrt{1-t^2}, \sqrt{1-t^2}[), \text{ for each real } t,$$

is the interior of the unit circle. (See Figure 7.) (For which values of t are the rectangles nonvoid? If t is restricted to $0 < t < 1$, how much of the figure remains?)

A natural description of this same graph is given in terms of trigonometric functions by

$$(]-\sin t, \sin t[,]-\cos t, \cos t[), \qquad 0 \leq t \leq \pi/2.$$

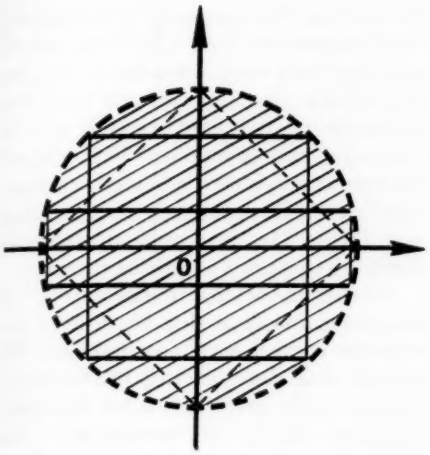


Figure 7

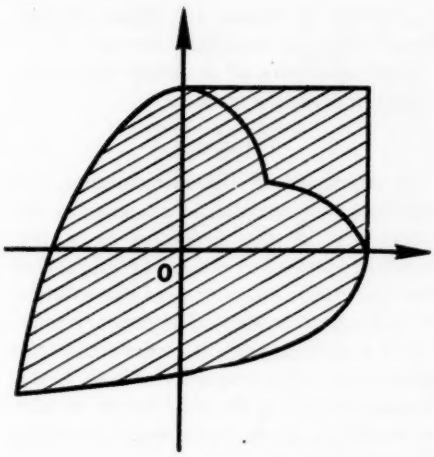


Figure 8

Example 4 shows quite naturally that a circular region is an infinite union of rectangles. (There need not be any "degenerate" rectangles, since the two degenerate cases may be replaced by a square with vertices at the four points otherwise excluded as shown in Figure 7.[2] Of course for the interior alone, the two degenerate cases may simply be omitted.)

A region such as that graphed in Example 4, in which the boundary is excluded, is called an open region. Is it possible to generate an open region by a set of *closed* rectangles? (See Figures 11*b* and 11*f*.)

That somewhat complex examples may have simple trigonometric descriptions is suggested by Example 5, while the types of graphs that may be obtained by the use of the absolute value function is suggested by Example 6. Example 7 illustrates the relation between exponential and logarithmic functions.

Example 5. The set of intervals

$$([\sin t, \cos 2t], [\sin t, \cos 2t]), \qquad -7\pi/6 \leq t \leq \pi/6$$

is graphed in Figure 8. Could t be restricted further and still yield the entire graph?

Example 6. Figure 9 shows the graph of the set of intervals

$$([t, |t|], [t, t+1]), \text{ for each real number } t.$$

Notice that the "handle" is generated by segments, while the triangular region is generated by nondegenerate rectangles.

Example 7. The set of intervals

$$([t, 2^t], [t, 2^t]), \text{ for each real } t,$$

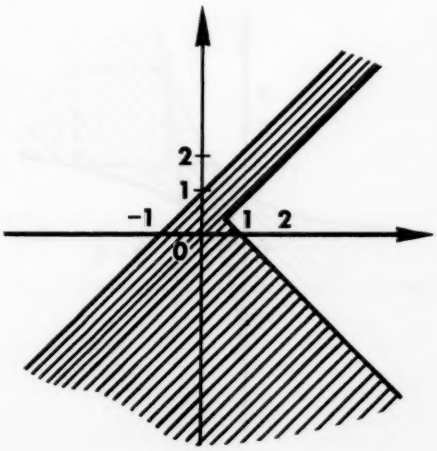


Figure 9

[2] In Figure 7 the circle is shown dashed to indicate that the boundary is not a part of the graph.

is graphed in Figure 10. What are the equations of the boundary curves?

Additional sets of intervals and their graphs appear in Figure 11.

Remarks

As suggested in the introduction, the principal consideration in graphing is not proficiency in the graphing of a few standard forms, but rather the realization that the graph is a picture of the solution set of an open sentence. The primary problem of graphing, then, is to determine the solution set to be graphed—at least to determine enough of the solution set so that an intelligent guess can be made regarding the rest of the elements.

As the examples illustrate, interval graphing provides fresh opportunities in finding solution sets in a vast number of differing problems at all levels of secondary mathematics. Many of the important ideas of mathematics can be incorporated into these problems providing preliminary introduction or fundamental review. The basic notions of set theory—union, subset, intersection, etc.—as well as the set theory notation can be introduced and employed to great advantage. (Modern set notation descriptions are even more concise and precise than the ones used in this paper.) The relations between a function and its converse has a simple description as a set of intervals. (See Figures 7 and 11*b*.[3]) The subtleties of the limit process can be illustrated and exemplified. The relations between trigonometric and algebraic functions can be brought out. (See Figure 4 and Figure 11*a*, in which trigonometric identities occur in a surprising manner.)

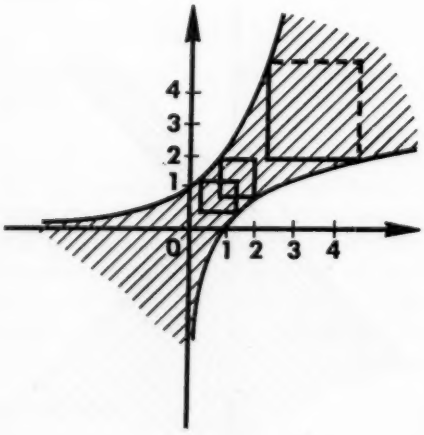
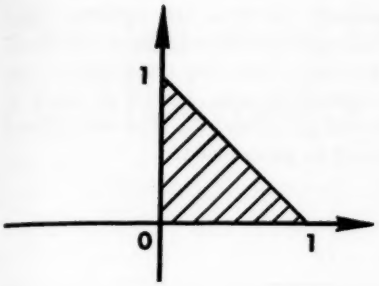


Figure 10

Another important "fringe benefit" of interval graphing is that the uses of parameters occur in a very natural way. The student soon discovers that the boundary curves of the graph are often described parametrically by the description of the set of intervals. Further experimentation will disclose to the student that parametric representation is not always unique, and that elimination of the parameters does not always give the curve described parametrically.

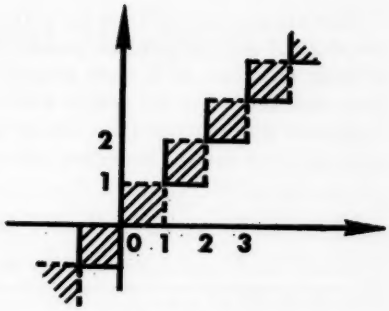
Interval graphing provides the student with an excellent opportunity to formulate and graph original examples. He should learn quickly that the equations of the boundary curves for the area generated can be determined by considering which sides (or vertices) of the generating rectangles determine the boundary, and conversely, he may learn how to formulate a set of rectangles to generate an area with given boundaries. With a minimum amount of graphing experience the changes that occur when closed intervals are replaced by open intervals, and vice versa, will become apparent, and the student also will be able to tell when the generating rectangles are all squares (possibly with some degenerate cases). He should try to determine conditions on the intervals so that the graph will be restricted to preassigned quadrants or regions.

[3] $[t]$ refers to the (unique) greatest integer less than or equal to t. Thus if t is an integer, then $[t]=t$, but whenever t is not an integer, $t-1<[t]<t$. "$y=[x]$" defines a function called "the greatest integer function." (The greatest integer function is commonly encountered as the "birthday function," that is, the relation between the time since a person's birth and that person's age.)



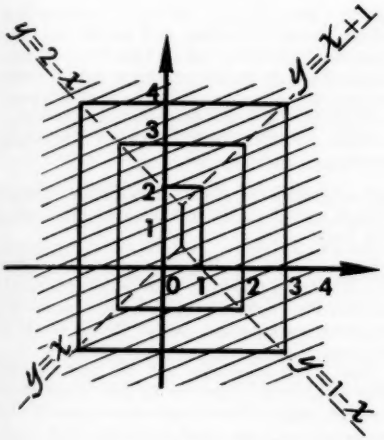
$([0, \cos^2 t], [0, \sin^2 t])$

Figure 11a



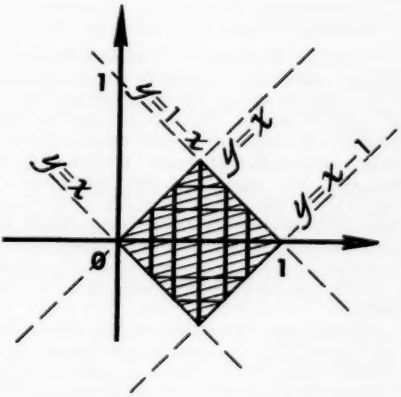
$([[t], t], [[t], t])$

Figure 11b[3]



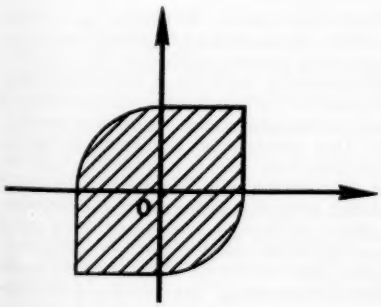
$([t, 1-t], [t, 2-t])$

Figure 11c



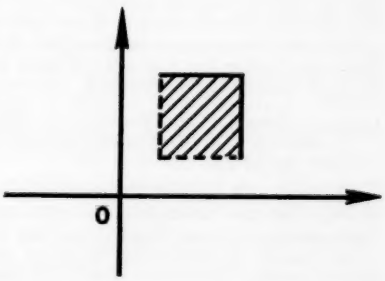
$([t, 1-t], [-t, t])$

Figure 11d



$([\sin t, \cos t], [\sin t, \cos t])$

Figure 11e



$([1+1/|t|, 3], [1+1/|t|, 3])$

Figure 11f

With appropriate aid from his instructor the student may attempt to answer more difficult questions of a more general nature, such as (1) For the case in which the endpoints are generated by linear functions of t, how can the boundary curves be determined just from the algebraic signs of the slopes of the endpoint functions? (2) Are there rules for translation, rotation, stretching, etc., similar to those of analytical geometry? (3) How can interval graphing be generalized?

Have you read?

COLEMAN, A. J., "Reforming the High School Mathematics Curriculum," *Canadian Mathematical Bulletin*, May, 1960, pp. 203–213.

This you must read, first because it will show you are not the only one with troubles, and second to see the things our neighbors to the north are thinking about. The problems are that gifted students say courses are dull, teachers of beginning mathematics are mostly English and physical education specialists, the number of fully qualified mathematics teachers is decreasing, the textbooks deal with inconsequential snippets, the only change in mathematics has been to water it down, and when grade 13 examinations test anything but memory they are called unfair.

What are Canadians doing about these problems? They have set up the Ontario Mathematics Commission which proposes a revision in grade 13 examination to measure competence in use rather than recall. They hope to break the barriers between algebra, geometry, and trigonometry, and to develop challenging aspects of the subject but keep the present core of content. Canadians work on the problems of textbook lag, lip service to modern mathematics, and the gulf between the mathematics professor and the high school teacher. Read this article and see how they propose to distribute the content, then judge for yourself.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

CONWAY, FREDA. "The Teaching of Statistics in Schools," *The Mathematical Gazette*, May, 1960, pp. 81–83.

Perhaps you haven't decided whether you should introduce statistics into your school's mathematics program. Read this article to find some reasons for introducing statistics: quantitative thinking, techniques of analyses, description of patterns of variation, and meeting the needs of science and social science.

What content will meet this need? The author believes that content should include such things as averages, frequency distributions, sampling procedures, indices, standardized numbers, mortality tables, and measures of dispersion. In high school all of this must be done with live data in an interesting manner. Under these conditions I am certain the author's point is well taken. Read this discussion and see what you think.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

RICE, CHARLES D. "Cupid with an Adding Machine," *This Week Magazine*, September 18, 1960, p. 27.

Here's a new one! Just in case you didn't know, it's math not love that makes the world go round. For example, what is the chance of a blind date working out 50-50? None at all. It takes more than two to make a match.

Then, after marriage, will the first born be boy or girl, are the chances 50-50? What about the second child? Have your students read this short article to put a spark in their study of statistics.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

ZANT, JAMES H. "The New World of Mathematics," *Mathematics Magazine*, March-April 1960, pp. 211–217.

Mathematics today is moving at a lively pace. New applicaitons are found in dealing with bacterial growth, radioactive life, and electronic calculating machines. These uses demand new number systems, more mathematics, a change in language, and continuous development. Two trends seem significant in mathematics programs, one is acceleration and the other is modernization. The first comes about by ability grouping, faster rates, and introducing algebra in Grade 8. The second trend means a more important role for concepts, definitions, precision, and continuity. What are the implications? A "new world" in content and efficiency means revisions, selection, direction, and co-operative effort for all concerned. We should read an article like this each week.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The Advanced Placement Program in Mathematics

RICHARD S. PIETERS, *Phillips Academy, Andover, Massachusetts, and* E. P. VANCE, *Oberlin College, Oberlin, Ohio.*

It is the great hope of those who have been involved in the Advanced Placement Program in Mathematics that teachers, no less than students, will find it an inspiration, a challenge, and a revitalization of their work.

IT IS WELL KNOWN THAT there is an increasing demand from our society for scientists and other personnel who are competent in mathematics. While this demand is not being met completely, something significant is being done in this direction. One of the most significant responses to this need is the Advanced Placement Program in Mathematics.

This country has been faced for some time with the important responsibility of developing more effectively the potentialities of students who have capabilities and interests in mathematics. More specifically, our country's immediate problem is one of spotting these students, showing them what mathematics really is, helping them to develop interest in mathematics, and giving them a program that will be effective in producing the competence desired.

Unfortunately, during the first half of the twentieth century, the content and organization of high school mathematics for college preparatory students in this country remained essentially unchanged. It is true that certain modifications of teaching methods and some minor changes in topical emphasis occurred as a result of the 1923 Committee on Mathematical Requirements and the 1944 Commission of Post-War Plans of the National Council of Teachers of Mathematics. But nothing really significant was the result.

However, the pressure had built up, and the recent developments and positive actions that have taken place in the 1952–60 period have not come as a surprise, however significant, important, and helpful they may be. What took place in this period was almost inevitable.

Before we specifically devote our attention to the Advanced Placement Program, mention should be made of some of these other recent developments in mathematical education. In this way we can see more easily the relationship between the other new programs and ours and, more important, the real value and significance of what the Advanced Placement Program has accomplished and its importance and promise for the future, in the framework of all the recent developments in the field.

The Commission on Mathematics of the College Entrance Examination Board started its work in 1955, under the leadership of Albert Tucker, chairman of the Department of Mathematics, Princeton University, and with the help of a grant from the Carnegie Corporation. Significant sponsorship of institutes for teachers of mathematics, both at the secondary-school and college levels, by the National Science Foundation, began in the summer of 1955. In 1954, Max Beberman received

a large grant from the Carnegie Corporation for his work in the Illinois Mathematics Project. In February, 1958, an invitational conference on the Identification and Education of the Academically Talented Student in the American Secondary School was held in Washington. The report of this group, sponsored by the Carnegie Corporation, has since been made public. In it the subcommittee in the field of mathematics made certain significant recommendations. In the summer of 1958, the first large session of the School Mathematics Study Group, under the leadership of Edward Begle of Yale University, began its work and writing on new curriculum and guides in mathematics at the level of grades 7 through 12. This was also underwritten by the National Science Foundation.

It should be noted, however, that already in 1952 the two studies which led most directly to the present Advanced Placement Program were getting under way. These were the study on "General Education in School and College" directed by Alan Blackmer of Phillips Academy, Andover, and the study on "School and College Study of Admission with Advanced Standing" under Gordon Chalmers of Kenyon College, both underwritten by grants from the Ford Foundation.

Mr. Blackmer's study was concerned with the great deal of evidence showing that many brilliant boys were bored either in the first year of college through repetition of material which they had already adequately covered in school, or were bored in the later years of their school work by repeating material which they had already assimilated but which their slower classmates had not mastered at the first attempt.

There also was evidence of fairly strong dissatisfaction with the current program in schools and colleges and many suggestions for changes were being made, not only changes which should improve the articulation between school and college programs, but also changes in the particular courses being given in both kinds of institutions. This was part of the stirring, and it soon led to the studies mentioned above.

The basic assumptions under which President Chalmers' group worked were:

1. The able student is wasting a lot of time in school.
2. The best place for a school boy or girl is in school.
3. The best teachers for boys and girls of this age are usually found in secondary schools.

The natural result of these assumptions was the feeling that more opportunity should be provided for these able students, but that it should be provided in the local school rather than by sending these youngsters to college at an early age. The difficulty naturally was that if this were to be done, courses would have to be provided that the colleges would accept, since they must be the equivalent of college work. Also some way would have to be found to convince the colleges that the students had adequately covered the suggested work.

In order to do this in the field of mathematics, a committee of scholars and teachers, under the leadership of Professor H. W. Brinkmann of Swarthmore College, was asked to outline a program which would be equivalent to that of one year of college work—and which could be used in high school. Of course, agreement by different colleges on such a program was extremely difficult to secure. The fact that any agreement was possible is undoubtedly due to the fact that many men realized that the situation indeed called for heroic measures.

The problem presented was not a simple one. New frontiers in mathematics were creating almost unlimited opportunities for growth in mathematical knowledge and applications to the physical sciences and engineering, to the social and biological sciences, and to business and industry. Instead of emphasis on the applications and

the utility aspect of mathematics at this level, stress was needed on the nature of the subject, its generality and open-ended quality, its wide, divergent fields, its system of logic and its continuous and lively growth. In order to accomplish this, the committee felt that, for the good student, the entire secondary-school curriculum could be redesigned as a three-year program, terminating in the "Advanced Placement Course" which would be one full year of college calculus with analytic geometry. (It should be noted that many of the changes sought by this committee have been adapted and adopted by the later studies of the Commission on Mathematics and the School Mathematics Study Groups in far wider connotations.) Although a year of calculus is not always the first year of college mathematics for a well-prepared student in a good college or university at the present time, a year of such mathematics does come into the college curriculum at some level. This will undoubtedly continue for many years to come.

When the first year of mathematics taught in the twelve original colleges of the School and College Study was considered, only four of these colleges covered precisely this material as a freshman course. Some also taught algebra and trigonometry. Others introduced considerable mathematical logic, methods of proof, and elementary set theory, before continuing with the calculus. The designed course, however, seemed to be the appropriate one and the most logical program to follow. The fact that such material actually constitutes one full year of college mathematics has made the problem of credit and placement a relatively easy matter. Some colleges were able to place the advanced placement students in their regular first-semester course in logic, from which the students would jump into the second-semester sophomore course, taking advantage of the college's special curriculum as well as the advanced placement course. In some cases, the students received only one semester of placement or credit, while in others they covered a year and a half of a specific college course. One fact was clear. When the individual college desired to work with these students, the details of placing them at the appropriate level were quite simple. No other one year of college mathematics would in any sense make this placement possible. Thus, the Advanced Placement syllabus has received almost universal acceptance and its present course description appears to be agreeable to all.

If such a program were to have a wide success, it would be necessary to persuade many schools to try it and to persuade many colleges that not only were schools trying it, but that the schools could succeed in giving courses which were indeed worthy of college credit. It is to the credit of many farseeing men in the colleges that they were able to persuade their colleagues to accept this experiment until it had proved its worthwhileness. One of the reasons for success in persuading colleges to accept the results of the program was the contribution made by many able and devoted men in setting up the examination, reading it, and grading the papers. Some of the leading college and university mathematicians of the country gave unstintingly of their time to assure that the syllabus was thorough, that the examination was searching, and that the grading was fair and at least as rigorous as that in their own courses on their own campuses. They could thus assure their sometimes doubtful colleagues and deans that they were not helping to lower their own standards, but rather to raise the sights of teachers and students all over the country.

Since the caliber of this program reflects the caliber of the men working on it, those who have served on the Advanced Placement Committee and given their time and effort should be mentioned. First is Professor Heinrick Brinkmann of Swarthmore College, the chairman of the committee that originally wrote the syllabus for the

program and, for three years, chairman of the examination committee and the readers' committee. Others are: Edward Begle of Yale University, now director of the School Mathematics Study Group; E. P. Vance of Oberlin College; George Thomas of M.I.T., director of the undergraduate program there and formerly member of the CEEB Commission on Mathematics; Garrett Birkhoff of Harvard; Gaylord Merriman of the University of Cincinnati; Rothwell Stephens of Knox College; and Albert Tucker of Princeton University and formerly chairman of the CEEB Commission. Each examination committee also has had representatives of secondary schools. These have included Carl Streams of the Mt. Lebanon High School in Pittsburgh; Harry Ruderman of the Hunter College High School in New York; William Matson, supervisor of mathematics in Portland, Oregon; Edward Walters of William Penn High School in York, Pennsylvania; and Richard Pieters of Phillips Academy, Andover, Massachusetts. Since Mr. Brinkmann's resignation from the committee in 1956, Mr. Vance has been serving as chairman of the Board of Readers and Mr. Pieters has been chairman of the Examination Committee.

The Examination Committee meets semiannually. The first meeting each year is at the Advanced Placement Conference in June. At this meeting preliminary policies are set, revisions of the syllabus are discussed, and agreement is reached as to the format of the examination. Then at a November meeting the examination is constructed. For the last few years it has consisted of thirty rather short questions of the multiple-answer type, for which a student is allowed one hour. Following are ten questions of the longer type, for which the student has to work out his own answers and present them on the examination paper. These are of such length and difficulty that two hours are allotted for their completion. In order to display the quality and coverage of the examination, the Educational Testing Service has given permission for the publication of a complete examination. It appears at the end of this article.

It should be noted that at the meeting of the Committee in November, 1960, it was decided to change the allotment of time so that the 1961 examination will consist of forty-five short questions for which one-and-a-half hours will be allotted and seven longer questions for which one-and-a-half hours will be allotted.

In recent years, not only because of opinions of the Examination Committee, but also because of expressed reactions from both secondary-school and college teachers, the test has been designed to pick students who are adequately trained in the equivalent of a full-year course in analytic geometry and calculus at a liberal arts college, and not just to "skim off the very top cream." In this respect the results have been most satisfactory. Although the examination has not been made easier for the best students, more relatively easy items have been included to avoid discouraging students and teachers.

An important part of the Committee's work, usually every other year, is to reconsider the syllabus as printed in *The Advanced Placement Program: Course Description*, the so-called Acorn Book of the College Board. In 1958 this syllabus was completely reworked in the light of experience with the program. In January of 1960 another revision was made.

Another obviously important part of the whole program is the reading of the examinations. The multiple-choice section is read by electronic machines, but the Part-II questions are read individually by a corps of readers under the chairmanship of E. P. Vance of Oberlin College. These readers are selected equally from colleges and from schools in the program.

The readers of the test have always been an extremely competent and efficient group. Despite the fact that the number of papers grew to 2908 in 1960, the group has worked as a unit, and each reader has had the opportunity to mark the entire test.

Thus the entire group of readers feels that they have had a significant part in the reading of the whole examination. Because it has been a team project, and because of the high qualities of the readers, the reading has resulted in a most consistent product.

The actual scoring of the tests is worked out as follows. Before the reading session, the chief reader works out the solution to each problem in the two or three probable ways. The steps in the solutions are noted and various amounts of credit assigned. On this basis sample examinations are scored and a revised partial credit system established. Before each problem is marked, the partial credits are discussed with the entire reading group, sample papers are scored and a group discussion leads to an agreement on the final partial credits for each problem.

During the first few years a comparison group of college students took this same examination, and a correlation of the grades of A, B, and C college students was used to check the validity of the grading.

The results of the grading are sent to the college at which the student has matriculated. These results consist of the separate scores on Part I and Part II and a composite score for the whole examination. The Part-II examination paper of each student is sent also to the college for its consideration in granting or withholding advanced placement and credit.

Most colleges and universities welcome advanced placement students in mathematics. Many give both credit and advanced placement. Some give advanced credit only. Very few which have admitted such students have not as yet set their policies. Fortunately, more colleges each year are giving credit, whether the student continues with his mathematics training or not. This is extremely significant. The colleges are doing this for two main reasons. They are aware of the high standard set by the program and are reasonably confident of the student's knowledge. Also, the colleges wish to encourage secondary schools and capable teachers at this level by recognizing publicly the college mathematics taught in secondary schools. Faith in the competent secondary-school teacher today is most important, especially when demonstrated by college admissions officers, registrars, and professors.

Although it is true that some colleges still question the standards of the program, this has almost always been done by those who are not too well acquainted with it and who are judging it on the basis of the performance of only a few candidates. This question will be discussed later in this article.

One measure of the significance of the Advanced Placement Program and its effect on mathematics education in secondary schools is simply the increase in number of those taking the examination.

Year	*Number of schools*	*Number of students*	*Number of exams in all subjects*	*Number of exams in mathematics*
1955	38	925	1,522	285
1956	110	1,292	2,475	380
1957	212	2,068	3,772	750
1958	360	3,717	6,804	1,177
1959	566	5,900	8,260	1,870
1960	890	10,540	14,200	2,908

As evidence of the quality of the work done in 1959, eleven hundred students got grades of 3, 4, or 5, which are grades good enough to get a full year's advanced standing and credit at many colleges including Harvard University.

Another measure of the significance of the Advanced Placement Program is the effect that it has on a school which goes into it wholeheartedly. The evidence is very strong that the effect can be electrifying. Not only are boys and girls inspired to do better work, but teachers find themselves challenged as never before, and the influence pervades the whole course. Definite ability grouping is increasing. We now find at least three distinct tracks in many of the large secondary schools. Problems

of each are discussed by the administration, the teachers, the parents, and the community. Each group—the advanced placement or honors group, the regular college-bound group and the noncollege-bound group—has benefited. The challenged teacher can now, with great benefit to the student, develop new methods for each. Better use is also made of the good teachers. In many cases, a well-prepared young teacher, who because of lack of seniority previously taught the ordinary beginning algebra and geometry, is now teaching the advanced placement section.

How can a school prepare for the Advanced Placement Course in Mathematics, which is a full college course in calculus and not an honors course in trigonometry or college algebra? There are many ways of doing it. One obvious way is that of straight acceleration, doing four years of mathematics in three, or, in the case of senior high schools, doing the work of three years in two. An increasing number of students are now having an algebra course in the eighth grade. If this is a good solid year of algebra they will be ready to proceed, not at an accelerated pace, but at a regular pace that will still bring them up to the threshold of their senior year having finished the standard secondary-school course in mathematics and ready for the calculus.

Many teachers have found that the mere availability of an advanced placement course is a strong inducement to ambitious students to work hard in their lower years so as to be able to take it when they become seniors. This probably is one of the greatest advantages to be derived from having such a program in a school.

We now come to an aspect of the program which is most vital. That is the question of the quality of teachers in the high schools that are attempting it. Many college instructors in mathematics are concerned that, with the pressures of the space age and keeping up with the Joneses, many schools are going into the Advanced Placement Program without giving their teachers adequate time and opportunity for preparing to teach this new course. Professor Neeley of Carnegie Institute of Technology has set forth this point of view very forcefully in an article he wrote in the *American Mathematical Monthly* some time ago. If calculus is to be taught in the schools as a rote exercise in manipulation it will be a serious disadvantage to the students who take it, rather than an advantage, when they get to college. They will have to unlearn and then relearn their material, and this is more difficult than learning it correctly in the first place. If teachers cannot teach some of the "why" of the calculus as well as the "how," they should not attempt it. That there *are* many teachers in schools of our country who can, is a most important point. This statement is strongly supported by evidence from many colleges where students have successfully continued a sequence of courses of which the first was an A.P. course in school. For instance, from the office of Advanced Standing at Harvard College: "We compared the grades of Sophomores in Mathematics 2A [third semester calculus] with those of Freshmen in the same course who presented Advanced Placement test scores of 3 or higher. To guard against the possibility of wide differences in mathematical aptitude, the MAT scores of the two groups were also compared. . . . The fact that the Freshmen in question do as well as, and in most cases, slightly better than, the Sophomores with comparable MAT scores, provides strong support for the validity of the Advanced Placement Examination. There is no evidence to support the view that the secondary school student introduced to calculus must possess extraordinary mathematical aptitude." What he must have is a perceptive, imaginative, and enthusiastic teacher.

Fortunately there are many opportunities to obtain some help for those who want to prepare to teach the Advanced Placement Program, and in the next few years there will be many more. Several of

the National Science Foundation Summer Institute courses are designed particularly for this purpose.

For anyone who is interested in the Advanced Placement Program and who wants to learn what other schools are doing, or how colleges are reacting, a very good idea would be to attend the Advanced Placement Conference. Such a conference has been held in June each year since 1955. At these conferences the experiences of various schools have been exchanged, the successes and failures of the program noted, and invaluable personal contacts made between school and college people.

Whether or not the Advanced Placement Program is right for your school is a question which only you and your department can decide in the light of the whole picture in your school and community. It should not be forgotten that this program is a program involving a whole year of calculus at a college level. If your school cannot get such a year's work in, it should not under any circumstances send its students up for the examination. This is unfair to them and to their teacher. This is not to say that a half year or less of calculus is not a good thing for some students, but it is an inadequate preparation for this particular examination. To convince people of this fact is part of the reason that the complete sample examination is printed with this article.

If, due to a reorganization of your mathematics curriculum, you have covered the amount of secondary-school mathematics which you feel is adequate, and you have somewhat less than a year's time left, you have to decide what to teach during this time. The College Board Commission has suggested work in probability and statistics which would normally take about half a year. The SMSG has recently published a suggested course in elementary matrix theory which would take the same length of time. There are many other suggestions which could profitably be put in at this point. Some people are experimenting with courses in finite mathematics, others with courses in concepts of mathematics using various texts and sometimes only their own lecture notes.

The Commission has said repeatedly that it feels that some work in probability and statistical inference is of great importance for those who are not going on in mathematics, as well as for those who are going on in mathematics, physics or chemistry, or other sciences. Its members feel that unless and until this has been finished one should not attempt the calculus. There is some disagreement with this point of view and many feel that once they have completed the courses in algebra, geometry, and trigonometry, even if they have only a half year, they would rather go directly into the calculus. It is undoubtedly true that there are more teachers who have had at least a minimum training in calculus than those who have had training in probability and statistical inference. This is probably one of the main reasons for this point of view. Also it is one reason why inadequate calculus courses have been mushrooming so rapidly in high schools that want to pay lip service to an advanced program.

Once again it should be emphasized strongly that the teaching of mere manipulation is as bad in the calculus as it is in algebra, if not worse. The A.P. syllabus is prepared with this in mind and examination questions are chosen to emphasize this idea, but not to overemphasize it, since we want our students to have facility as well as understanding. Calculus, if taught at all, must be taught for meaning and understanding. If it is not so taught it becomes but a series of memorized formulas. If it is, it can open the gates to greater understanding, not only of mathematics, but of all the physical and social sciences. It is this opening of the gates of a student's mind which is the objective of the Advanced Placement Program no less in mathematics than in history and English. It is the great hope of those who have been involved in the program in mathe-

matics that teachers, no less than students, will find it an inspiration, a challenge, and a revitalization of their work. . . . Those who have made the attempt have almost universally found it so. We hope that many more teachers will want to accept the challenge to prepare themselves adequately, if they are not so prepared, and to lead students to conquer new fields.

REFERENCES

1. *Advanced Placement in Independent Schools.* Independent Schools Association, 84 State Street, Boston 9, Massachusetts.
2. *The Advanced Placement Program: Course Description.* CEEB, 475 Riverside Drive, New York 27, N.Y.
3. *Bulletin of National Association of Secondary School Principals.* December, 1958. NEA, 1201 16th Street, N.W., Washington 6, D.C.

Advanced Placement Examination, Mathematics Section I, 1958

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MATHEMATICS, SECTION I
TIME—1 HOUR

Directions: Solve the following problems, using the blank pages for scratchwork. Indicate your answers on the answer sheet. No credit will be given for anything written in the examination book. Do not spend too much time on any one problem.

1. The equation of the circle with center $(2, -1)$ and radius 3 is

   (A) $x^2-4x+y^2-2y=4$
   (B) $x^2+4x+y^2-2y=4$
   (C) $x^2-4x+y^2+2y=4$
   (D) $x^2-4x+y^2+2y=14$
   (E) $x^2+4x+y^2+2y=14$.

2. The equation of the line whose intercepts are twice those of the line $2x-3y-6=0$ is

   (A) $2x-3y-3=0$
   (B) $4x-6y-12=0$
   (C) $4x-6y-6=0$
   (D) $2x-3y+12=0$
   (E) $2x-3y-12=0$.

3. A relative maximum value of the function $y=x^3-\frac{3}{2}x^2+2$ is

   (A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$ (E) 2

4. If $y=\frac{2}{4+x^2}$, then $\frac{dy}{dx}=$

   (A) $\frac{2}{(4+x^2)^2}$ (B) $\frac{2x}{4+x^2}$ (C) $\frac{4x}{(4+x^2)^2}$
   (D) $\frac{-4x}{(4+x^2)^2}$ (E) $\frac{-2x}{(4+x^2)^2}$.

5. The distance from the point $(2, 2)$ to the midpoint of the segment joining $(2, 3)$ with $(-4, -1)$ is

   (A) $2\sqrt{2}$ (B) 3 (C) $\sqrt{10}$
   (D) $3\sqrt{2}$ (E) $\sqrt{13}$.

6. The value of k that makes the pair of lines $\begin{matrix}6x-9y=5\\ kx-4y=8\end{matrix}$ perpendicular is

(A) -6 (B) $-\frac{8}{3}$ (C) $\frac{3}{8}$

(D) $\frac{8}{3}$ (E) 6.

7. The number of points of inflection of the curve $y=x^5+5x^3+10x+1$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4.

8. The area between $x^2y=1$ and the x-axis from $x=\frac{1}{2}$ to $x=1$ is

(A) $\frac{7}{24}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{5}{3}$

(E) 2.

9. The radius, x, of a sphere increases with the time, t. What is the radius of the sphere when the surface area ($S=4\pi x^2$) and radius are increasing at the same rate?

(A) $\frac{1}{8\pi}$ (B) $\frac{1}{\pi}$ (C) π (D) 2π

(E) 8π.

10. $\int_1^e \frac{dx}{x}=$

(A) 0 (B) 1 (C) e (D) $e-1$

(E) $\log_e x$.

11. $\int_2^5 \frac{dx}{\sqrt{x-1}}=$

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2 (E) 3.

12. The slope of the curve $x^2+2xy+3y^2=3$ at the point $(2, -1)$ is

(A) -2 (B) -1 (C) 0

(D) 1 (E) 2.

13. An equation of the line tangent to the curve $y=\sin 2x+3\cos 2x$ at the point on the curve where $x=0$ is

(A) $2x-y=-3$ (B) $2x-y=-1$

(C) $6x+y=1$ (D) $2x+y=3$

(E) $x-2y=-3$.

14. The equation of the parabola whose focus is $(-1, 3)$ and directrix is the line $y=-1$ is

(A) $x^2+2x=8y$ (B) $x^2+4x-y=-6$

(C) $24x^2=8y$ (D) $2x^2+5y=17$

(E) $x^2+2x-8y+9=0$.

15. A point moves along the x-axis according to the law: $x=3t^2-t^3$. When the acceleration is zero, the velocity is

(A) 0 (B) 1 (C) 2 (D) 3

(E) none of these.

16. The derivative of $\log_e(x^5)$ is

(A) $5(\log_e x^4)$ (B) $5x^4(\log_e x)$

(C) $5x^3$ (D) $5x$ (E) $\frac{5}{x}$.

17. If $x=2t-3$ and $y=4-5t^2$, then $\frac{dy}{dx}$ is

(A) $-20t$ (B) $-10t$ (C) $-5t$

(D) $-5x$ (E) $-5(x+3)$.

18. The second derivative of $f(x)=e^x+e^{-x}$ is

(A) e^x-e^{-x} (B) $2f(-x)$ (C) $f(x)$

(D) $-f(x)$ (E) $x^2f(x)$.

19. The area bounded by the parabola $x=2y^2$ and the line $x=8$ is rotated around the x-axis. The volume of the solid of revolution swept out is

(A) 8π (B) 10π (C) 12π (D) 14π

(E) 16π.

20. $\int_0^1 xe^{x^2}dx$ is equal to

(A) e (B) e^2 (C) $\frac{1}{2}(e-1)$

(D) $2e$ (E) 1.

21. $\lim_{x\to 0} \frac{\tan 2x}{x}$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) ∞.

22. At what point on the curve $y=\sqrt{4-x}$ is the tangent parallel to the line $x+2y=7$?

(A) $\left(\frac{63}{16}, \frac{1}{4}\right)$ (B) $(3, -1)$

(C) $(0, 2)$ (D) $(3, 1)$

(E) $(4, 0)$.

23. One of the following statements is false with regard to the graph of $y=\frac{x-1}{x^2(x-2)}$. Which one is false?

(A) The x-intercept is 1.
(B) There are two vertical asymptotes.
(C) There is one horizontal asymptote.
(D) There is no y-intercept.
(E) The graph is symmetric to the y-axis.

24. The minimum value of $y=2\sqrt{x-1}$ occurs at

(A) $x=-1$ (B) $x=0$ (C) $x=1$

(D) $x=2$ (E) none of these.

25. The area under the curve $y=\tan x$ between $x=0$ and $x=\frac{\pi}{3}$ is

(A) $\log_e 2$ (B) $\cot\left(\frac{\pi}{3}\right)$

(C) $-\cot^{-1}\left(\frac{\pi}{3}\right)$

(D) $-\log \sin\left(\frac{\pi}{3}\right)$ (E) $\frac{1}{2}$.

26. If $f(x)=(1-x)^3$ and $g(x)=\frac{1}{x}$, then the derivative of $f(g(x))$ is

(A) $3(1-x)^{-4}$ (B) $3\left(-\frac{1}{x^2}\right)^2$

(C) $3(1-x)^2$ (D) $3\left(1-\frac{1}{x}\right)^2$

(E) none of these.

27. The area bounded by the curve $x=(y-1)^2$ and the straight line $x=y+1$ is

(A) $\frac{3}{2}$ (B) $\frac{9}{2}$ (C) 9 (D) $\frac{27}{2}$

(E) 27.

28. The curve $y=f(x)$ between the points where $x=a$ and $x=b$ is revolved about the line $y=1$. If $f(x)>1$ for $a\leqq x\leqq b$, the volume of the solid generated is

(A) $\pi\int_a^b [f(x)]^2dx$

(B) $\pi\int_a^b ([f(x)]^2-1)dx$

(C) $\pi\int_a^b [(f(x))^2-2f(x)+1]dx$

(D) $\pi\int_a^b (f(x)+1)^2dx$

(E) none of these.

29. If $y=x^2-x$, the derivative of y^2 with respect to x^2 is

(A) $2x-1$ (B) $2x^2-3x+1$

(C) $4x^3-6x^2+2x$ (D) 2

(E) $12x^2-12x+2$.

30. The area in the first quadrant next to the y-axis and bounded by that axis and the curves $y=\cos x$ and $y=\sin x$ is rotated about the x-axis. The volume of the solid generated is

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{\pi}{4}-\frac{1}{2}$

(D) $\frac{\pi^2}{2}$ (E) $\frac{\pi^2}{4}-\frac{\pi}{2}$.

Advanced Placement Examination, Mathematics Section II, 1959

MATHEMATICS, SECTION II
TIME—2 HOURS

1. Find the area of the region bounded by the curves $y=x-2$ and $y=2x-x^2$.
2. Consider a parallelogram and any point in its plane. Prove by analytic geometry that if the sum of the squares of the distances of the point from two opposite vertices of the parallelogram is equal to the sum of the squares of the distances of the point from the other two vertices of the parallelogram, then the figure is a rectangle.
3. The area in the first quadrant bounded above by the curve $y=\sin x$, below by the x-axis, and on the right by the line $x=\frac{\pi}{2}$ is divided into two equal parts by the line $x=c$. Find c.
4. A light is on top of a pole h feet high. A ball is dropped from a point at the same height as the light but k feet horizontally away from it. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ second later? Assume that the ball falls a distance $s=16t^2$ feet in t seconds.
5. Discuss the graph of $y=(x^2+1)e^x$. Consider
   (a) the intercepts
   (b) asymptotes
   (c) maxima and minima
   (d) points of inflection
   (e) behavior for large $|x|$
   (f) symmetries.

   Choose convenient scales and SKETCH THE CURVE. (See figure at top of second column.)

6. The graph of $y=\dfrac{ax+b}{(x-1)(x-4)}$ has a horizontal tangent at the point $(2, -1)$. Find a and b, and show that the function has a relative maximum at this point.
7. Given a function $f(x)$ such that $f(1)=f(2)=4$, and such that $f''(x)$ exists and is positive throughout the interval $1\leqq x\leqq 3$:
   (a) What can you say about the sign of $f'(3)$?
   (b) Prove your statement, stating whatever theorems you use in your proof.
8. A straight fence 100 yards long stands on a ranch. The fence is to be left standing, and part or all of it is to be used in forming a rectangular corral, using an additional 260 yards of fencing for the other three sides. Find the maximum area which can be so enclosed.
9. The area bounded by the curve $y=e^{-x}$ and the lines $y=0$, $x=0$, and $x=10$ is rotated about the x-axis. Compute to three significant figures the volume of the solid of revolution so generated. Justify any approximations which you use.
10. $P(p, q)$ is a variable point on the parabola $y=x^2$. $V(0, 0)$ is the vertex of the parabola. $R(0, r)$ is the point where the perpendicular bisector of VP intersects the y-axis. Find $\lim_{p\to 0} r$.

HOBLADIC

GEORGE W. HORTON, *Illinois College, Jacksonville, Illinois.*

Here is a computer that you and your class can build at no cost.

HOBLADIC? Most of the high speed digital computers of today are known by coined names. This one was coined to designate HOrton's BLAckboard DIgital Computer. It cannot correctly be called a high speed computer because it is designed not for speed but for ease of understanding.

This paper describes how a HOBLADIC can be constructed by any student or teacher. The HOBLADIC can then be used to illustrate the basic theory and programming of the modern digital computer—without wires, switches, relays, magnetic drums, or similar equipment.

COMPUTERS IN GENERAL

Practically all digital computers operate, in the abstract, in the same manner:

1. All of them have a *memory*, or storage system.
2. *Data*, i.e., numbers for use in the projected computation, can be fed into (read-in) the machine in a section of the storage reserved for that purpose.
3. *Instructions*, coded, can be given to the machine to perform certain arithmetic operations with the numbers already in the memory, the answers being stored in designated memory cells. An instruction usually tells where the next instruction is to be found.
4. *Answers* may be extracted (read-out) from a specified reserved section of the memory.
5. All possible operations are reduced to simple combinations of addition, subtraction, multiplication, and division.* These operations are performed by a section of the computer which may well be called the *arithmetic unit.*
6. The output of the arithmetic unit at any time is stored in a section variously known as the *accumulator* or *register*, essentially an adjunct of the arithmetic unit.

* A study of the circuitry of an electronic computer would bring out the fact that these four processes are further reducible to the single one of addition; this paper is not concerned with circuitry.

SPECIFIC APPLICATION

HOBLADIC abstracts the above items and functions and makes certain substitutions and adaptions which are visible and understandable to all. Two of these merit special mention.

1. *Memory:* The popular press has made a fantastic fetish of machine "memory." Actually, the memory function consists of producing some physical change which will remain fixed until it is removed by positive, intentional action. The binary system of numbers enters here because it is based on the two alternatives of "1 or 0," "Yes or No," "bent or straight," "is or ain't," and so on. Many of the sophisticated computers use as memory mechanisms steel elements that are either magnetized or not, but a chalk mark on a blackboard is a perfectly satisfactory memory device. It has all the necessary attributes. It may be placed on the board, left there as long as it is needed, and then erased to make space for a new number.
2. *Arithmetic unit:* The mind and hand of the operator constitute the arithmetic unit of HOBLADIC. The operator needs only to be able to read and write, and to add, subtract, multiply, and divide. (The computer has been successfully operated by a fifth-grade boy.)

The computer in its simplest form is merely a blackboard, in one instance 3 by 4 feet, divided into five general sections: (1) the accumulator, (2) the read-in section of memory, (3) the read-out section of memory, (4) the remainder of memory, and (5) the arithmetic unit. (See Figure 1 below.)

The accumulator section is a panel on which numbers may be written. In many of the operations, the presence of a number in the accumulator may be taken for granted. Actually, the number to be placed there is either the starting or finishing number involved in an arithmetic process and will be displayed automatically in the arithmetic unit.

It will be assumed that there are 50 memory cells, numbered from 00 to 49. There is nothing unique or critical about this number; it could be more or less. The read-in section, then, will consist of two strips numbered respectively 00–04 and 05–09, in which data may be written. In actual operation, it is found that many operations are simplified if the two strips, 00–04 and 05–09, are removable. (Hooks on the main board and holes in the strips make this practical.) The strips may be called data cards.

Memory cells, 40–44 and 45–49, are read-out strips, made in the same manner as the read-ins, 00 and 05.

Specifically displayed intermediate storage consists of two vertical banks of five marked squares each, so marked that each may be given its own designator number when the occasion arises and a computation-involved number written in.

The arithmetic unit is a bare blackboard.

There is nothing unique about the geometrical distribution or orientation of the various compartments except that the arrangement of the read-in and read-out sections as horizontal strips and the intermediate storage as a vertical display has seemed to help in the understanding of the device.

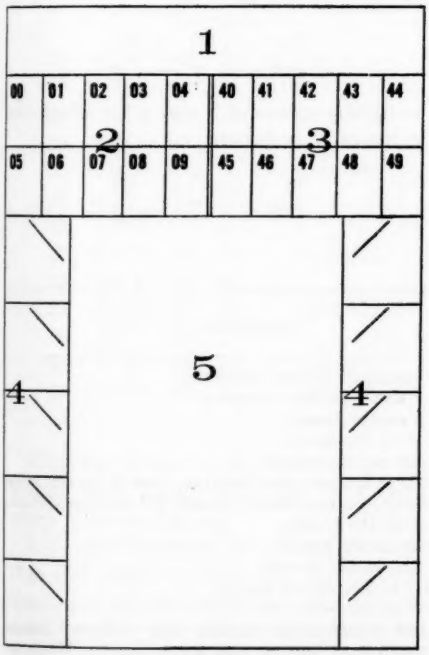

Figure 1

Instruction coding

The instructions are given by means of a three-letter code, essentially cognate. This could be made more consistent with machine operation if each instruction was given a two-digit number code, but this seems hardly necessary. The following will be sufficient for the programming of many illustrative examples:

ADC—Accept Date Card _____. This card will be either 00 or 05. In either case, the other four entries on the same strip are accepted at the same time and become displayed and available.

ADD—ADD to the number in the accumulator, or on the board, called the "board number," the number now in location _____.

DIV —DIVide the board number by the number now in _____.

DLC—DeLiver the answer Card 40 (41, 42, 43, 44) or 45 (46, 47, 48, 49).

EWB—Erase and Write on the Board the number now in _____.

MPY—MultiPlY the board number by the number now in _____.

NZO—Non-ZerO. Test the accumulator (or board). Is it zero? This special instruction will be considered later.

SNH—Store the Number in Hand, i.e., the last number used in the program of computation, in _____.

STB—STore the Board number, the last one written in the arithmetic unit, in memory cell _____.

SUB—SUBtract from the board number the number now in _____.

Instruction cards

These cards as a group contain the program. Individually, each contains: (1) the number of the cell containing the instruction; (2) the instruction, in code, the operation to be performed; (3) the location of the number to be used in the operation; (4) the location of the next instruction. HOBLADIC instruction cards will thus consist of six digits and three letters, e.g., "10 ADC 05 11" interpreted, says, "This is the instruction stored in cell 10: Accept the data card 05 (display it by hanging it on the proper hooks) and go to cell 11 for the next instruction." Similarly, "18 EWB 39 19" interpreted, says, "18: Erase, write on the board the number now in 39; next instruction in 19." Pupils soon become adept at these interpretations.

In one case, the HOBLADIC instruction is a "branching instruction." There are several instructions of this type in the big machines. In this case, the last two pairs of digits are both instruction addresses. Thus, "21 NZO 12 33" interpreted, says, "21: Test the accumulator (or board) for *zero*. If it is not zero, go to 12 for next instruction. If it is zero, go to 33 for next instruction."

Incidentally, it should be noted that the computer will stop if it has no further instruction or if it is given an impossible instruction.

Some typical programs

Example 1. Given various values of x and y, find $N=2x^2+3xy$.

In $\cdots 00 \cdots 01$
put $\cdots x \cdots y$

using the values of x and y for which the value of N is desired;

in $\cdots 05 \cdots 06$
put $\cdots 2 \cdots 3$

TABLE 1

Instruction address	Instruction		Next instruction	Operation
10	ADC	05	11	Place constants in the computer.
11	ADC	00	12	Place x and y in the computer.
12	EWB	00	13	Place x on the board.
13	MPY	00	14	Place x^2 on the board.
14	MPY	05	15	Place $2x^2$ on the board.
15	STB	30	16	Place $2x^2$ in intermediate storage. One of the side compartments is numbered 30 and $2x^2$ written therein.
16	EWB	06	17	Place 3 on the board.
17	MPY	00	18	Place $3x$ on the board.
18	MPY	01	19	Place $3xy$ on the board.
19	ADD	30	20	Add $2x^2$ to $3xy$ on the board.
20	STB	40	21	Place N in read-out cell.
21	DLC	40	11	Read-out N and cycle, starting over with new values of x and y as long as any are available. When no more data cards are available the process stops.

which are constants to be used in the computations.

Place the following program of instruction cards in the hands of a "caller" (or several callers if group participation is desired). This is equivalent to reading-in the instructions, and the instructions may be considered as in the computer in their numbered cells. Turn control of the computer over to 10. The program continues as shown in Table 1, which appears on facing page 214.

The example is actually too simple. If one read in 20 pairs of x and y, one would get 20 answers, but they could not be identified! The correction of this fault will be brought out in the next program.

Example 2. Given x, y, z, and p, find $q=xy+zp$.

The data card reads:

00	01	02	03
x	y	z	p

Give control of the computer to 13.

Program

13	ADC	00	15	
15	EWB	00	23	
23	SNH	40	25	· · · (x in 40)
25	MPY	01	35	
35	SNH	41	16	· · · (y in 41)
16	STB	39	22	
22	EWB	02	17	
17	SNH	42	33	· · · (z in 42)
33	MPY	03	27	
27	SNH	43	18	· · · (p in 43)
18	ADD	39	21	
21	STB	44	12	· · · (q in 44)
12	DLC	40	13	· · · delivers card and cycles

Thus the answer, q, appears on the delivered card along with the data from which it was computed.

It is also brought out in this example that the order of *numbering* the instructions need not be sequential; the requirement is that the "next instruction address" be such that the *operations* themselves are properly sequential.

Example 3. The building of a table involving a finite number of definite increments in one of the variables, a process which makes use of the instruction, NZO, is illustrated by the old favorite, the Towers of Hanoi. In this problem, the number of valid moves to tranfer one pile to another of the three available locations is given by $M=2^n-1$, where M is the number of moves required, and n is the number of discs in the original pile. The problem, then, is to compute

$$M=2^n-1, \quad \text{for} \quad 1\leqq n\leqq 64, \quad \triangle n=1.$$

The data card reads:

00	01	02
1	2	64

Read-in the program; transfer control to 10.

Program

(Setting up)

10	ADC	00	11
11	EWB	01	12
12	STB	38	13
13	SUB	00	14
14	STB	41	15
15	EWB	00	16
16	STB	39	17
17	STB	40	18
18	DLC	40	19

(Routine cycle)

19	EWB	39	20
20	ADD	00	21
21	STB	39	22
22	STB	40	23
23	EWB	38	24
24	MPY	01	25
25	STB	38	26
26	SUB	00	27
27	STB	41	28
28	DLC	40	29

(Test for completion)

29	EWB	02	30
30	SUB	39	31
31	NZO	19	88

If operation 31 does not give zero, the next address is 19; if it is zero, the next address is 88, a nonexistent address, and the computer will stop.

It may take considerable time to solve the traditional problem of the 64 discs, $n=64$, but the complete process can be illustrated satisfactorily by changing the 64 to a 4 and being satisfied with a much smaller table!

The expansion of HOBLADIC is quite inexpensive. If one cares to use a computer with 100 memory cells, all that is required is to make the statement that 100 cells are available. Your computer thus proves to be an adaptable device.

With added cells it becomes quite possible to expand the third example to include in the table the *number of moves yet to be made* at the end of any integral completion for n.

Conclusion

After working through the illustrated examples, students may be encouraged to choose their own problems and do their own programming. There is no better way than this to illustrate the need for breaking up a problem into individual steps. At the same time, it should become quite clear that the machine does no thinking for itself; it will do exactly and only what it has been instructed to do.

The author has found the image set up by HOBLADIC to be of immense value to him as he approached the operation of three different commercial computers. They are all alike in essence.

Consecutive integers from 1961

The object is to express the various integers by using only the four digits 1, 9, 6, 1, standard mathematical symbols and no other numbers. (!3 = sub-factorial 3 = 2.)

$1=-1+9-6-1$	$21=1\cdot\sqrt{9}(6+1)$	$41=1+(6-1)!/\sqrt{9}$
$2=1\cdot9-6-1$	$22=11(6/\sqrt{9})$	$42=-19+61$
$3=1+9-6-1$	$23=(1+\sqrt{9})\cdot6-1$	$43=9\cdot6-11$
$4=1\cdot9-6+1$	$24=(1+\sqrt{9})\cdot6\cdot1$	$44=-1+9(6-1)$
$5=1+9-6+1$	$25=(19+6)\cdot1$	$45=1\cdot9(6-1)$
$6=6(\sqrt{9}-1-1)$	$26=19+6+1$	$46=1+9(6-1)$
$7=-1+\sqrt{9}+6-1$	$27=9\cdot6/(1+1)$	$47=16\sqrt{9}-1$
$8=-1+\sqrt{9}+6\cdot1$	$28=(1+\sqrt{9})(6+1)$	$48=(-1+9)\cdot6.1$
$9=1\cdot\sqrt{9}+6\cdot1$	$29=6\sqrt{9}+11$	$49=(-1+9)\cdot6+1$
$10=1\cdot\sqrt{9}+6+1$	$30=(6+9)(1+1)$	$50=(1+9)(6-1)$
$11=1+\sqrt{9}+6+1$	$31=1\cdot\sqrt{961}$	$51=(16+1)\sqrt{9}$
$12=9+6/(1+1)$	$32=(9-1)\sqrt{16}$	$52=-1+9\cdot6-1$
$13=(19-6)\cdot1$	$33=11(9-6)$	$53=1\cdot9\cdot6-1$
$14=1\cdot9+6-1$	$34=(16+1)(!\sqrt{9})$	$54=1\cdot9\cdot6\cdot1$
$15=1+9+6-1$	$35=19+16$	$55=1\cdot9\cdot6+1$
$16=1\cdot9+6+1$	$36=9(6-1-1)$	$56=1+9\cdot6+1$
$17=1+9+6+1$	$37=1+9\sqrt{16}$	$57=-1-\sqrt{9}+61$
$18=1\cdot\sqrt{9}\cdot6\cdot1$	$38=19\sqrt{\sqrt{16}}$	$58=-1\cdot\sqrt{9}+61$
$19=1\cdot\sqrt{9}\cdot6+1$	$39=6^{(1+1)}+\sqrt{9}$	$59=1-\sqrt{9}+61$
$20=1+\sqrt{9}\cdot6+1$	$40=(-1+9)(6-1)$	$60=(1+9)\cdot6\cdot1$

Many of the integers can be expressed in a variety of other ways. In some cases, the neatest representation was discarded in favor of one in which the digits 1, 9, 6, 1 appear in that order. Sub-factorial 3 = !3 = 2. The reader may care to extend the series.—Charles W. Trigg, *Los Angeles City College.*

Digital computer programming in high school classes

LOWELL T. VAN TASSEL, *Herbert Hoover Senior High School, San Diego, California.*

A student can obtain programming experience without an electronic computer.

THE WRITER OF AN ARTICLE in a recent issue of the *American Mathematical Monthly* quoted Professor Perlis to the effect that "whereas we *think* we know something when we learn it, and are *convinced* we know it when we can teach it, the fact is that we don't *really* know it until we can code it for an automatic computer."[1]

Last year, in two "upper track" (i.e., advanced) eleventh-grade classes at Hoover Senior High School, San Diego, California, class and teacher embarked on an unusual and ambitious program. We spent about a week examining in some detail the nature of the real and complex number system. The following week we attempted to relate our findings to applications in industry, and in particular to the programming of digital computers.[2] Vocational opportunities in this field also were explored.

With some encouragement, students first of all set up the model computer[3] essentially as shown in Figure 1. The model computer can be conceived as made up of two main units: a memory bank and an operation or computation section.

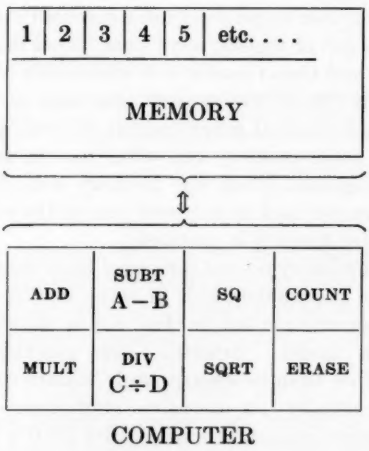

Figure 1

We assume that the "Add" section merely finds the sum of all addends that enter it, and will read out to any memory cell on direction. Arbitrarily, we say an "Erase" signal is necessary to zero "Add." "Mult." produces products of all factors entered, and will likewise read out as commanded. We assume for "Subt." that this unit produces $A-B$, where A is entered first, and then B, in that order. Again arbitrarily, "Div." produces quotients of C divided by D, the order of entries being first C, then D.

[1] George E. Forsythe, "The role of numerical analysis in an undergraduate program," *American Mathematical Monthly*, October 1959, p. 656.

[2] The slide rule, as an *analog* computer, should be noted here, and contrasts and comparisons made, especially after the unit is completely finished and the students understand the great difference in details.

[3] A book of general usefulness to the teacher here is *Electronic Digital Computers* by Franz L. Alt (New York: Academic Press, Inc., 1958), pp. x+336.

"Square" and "Square root," especially the former, are for convenience, as they make programming easier for the beginner. The "Count" register is handy for computing means—as addends are sent to "Add" while n, the *number* of counts, is kept in "Count." An arithmetic mean can be produced in "Div." by first sending in total found in "Add" (C in previous notation) and then sending in "Count" (D in previous notation). The mean then can be read out to any convenient memory cell. Also, considerably more sophisticated logical decision processes can be accomplished by exploiting the "Count" register.

The coding system is of the "from-data-to" type. Each line of the program tells the computer to get *from* some specified memory cell or register some *data* stored therein, and then transfer it *to* some other location. One of the ground rules used in our class was that every operation result goes to some memory cell after having been computed. From the memory cell, it is then fed back to any new part of the computer where it is required.

Probably no real computer looks exactly like the model, and, in most computers, segments are not as clear cut as shown in the model. "Squares" are convenient, rather than necessary; and "square root" is usually an iterative subroutine—an approximation process, rather than a discrete operation cell. "Erase" is a signal sent to a unit to clear any pre-existing data in addressee cells. The memory is considered to be of sufficient size, as far as number of cells are concerned. No limit was placed on the length of numbers that could be stored in each memory cell.

Our first problem on the digital computer involved the writing of a program (adding-machine tape could be used here for dramatic effect) for the solution of, say, $2x+5=17$, or, more generally, $ax+b=c$. The purpose of this elementary problem is to make the student focus on the capabilities and limitations of the model computer. The computer will do only and exactly what it is told to do. The student also perceives that one might just as well write the general solution to begin with. The purpose of "Erase" and uses of memory also become apparent to the novitiate program writer. Certain niceties, such as leaving a "clean" machine and having the answer appear in a given memory cell, can receive minor emphasis.

The student should think approximately as follows for the solution of $ax+b=c$: Clear all memory cells and operations registers. Enter data into memory. For example, enter a in memory 1, b in memory 2, c in memory 3. (The student should observe that the x itself does not appear in the program.) From memory 3 enter c in "Subt." From memory 2 enter b in "Subt.," it being understood that this sequence of entries produces $c-b$, in "Subt." Read $c-b$ from "Subt." into memory 4 and then from memory 4 read data into division operation section. Now from memory 1 read a into division cell, understanding that this sequence produces $(c-b)/a$. Read out answer to, say, memory cell 5. Erase all save memory 5.

More complicated linear problems in real as well as single complex variables should now be attempted. I encouraged students to make up their own command instructions, and to modify the computer if they saw fit. It is instructive to program, for example, $2x+3iy=5+7i$ and also to program a general solution for two linear equations in two unknowns. Here, after erase signal, one may have a tree or branch circuit. In the event that the determinant of coefficients is zero, program stops. In the event of a nonvanishing determinant, program continues.

A test programming problem to see how well this material has been learned is to have students program a solution to the general quadratic equation, under the assumption that both roots are real. An original program, Figure 2, by a student at Hoover shows his attempt. It is instructive to follow his solution through in detail, and then to attempt other solutions that approximately parallel his program.

Problem

Given the Quadratic Equation:

$Px^2+Qx+R=0$, with $Q^2-4PR \geqq 0$.

Find: The two roots.

Program:

1 E	14 tm6 sa	27 tm5 f
2 eP m1	15 tm7 sb	28 tm1 f
3 eQ m2	16 ts m8	29 tf m13
4 eR m3	17 tm8 sqrt	30 tm11 dc
5 e4 m4	18 tsqrt m9	31 tm13 dd
6 e2 m5	19 tm2 sb	32 td m14
7 tm2 f	20 tsb m10	33 tm12 dc
8 tm2 f	21 tm10 a	34 tm13 dd
9 tf m6	22 tm9 a	35 td m15
10 tm4 f	23 ta m11	36 tm14 c bp
11 tm1 f	24 tm10 sa	37 tm15 c bp
12 tm3 f	25 tm9 sb	38 Em14
13 tf m7	26 ts m12	39 Em15

Symbols used:

E	Erase
e	Enter
t	Transfer
m	Memory unit (usually followed by number)
sa	Subtract A
sb	Subtract B
s	Subtract $(A-B)$
a	Add
f	Multiply
sqrt	Square root
dc	Divide C
dd	Divide D
d	Divide (D)
p	Photorelay*
b	trip*
c	reader*
g	test problem

* Steps p, b, and c involve a Polaroid-Land camera taking a picture of the digital voltmeter display—an invention of the student programmer, for making a permanent record of final answer for files.

Figure 2

We concluded our work in digital programming with the solution to the general quadratic. However, in advanced twelfth-grade classes, Newton's method for real roots of polynomials could be programmed, as well as any special tabulation of functions that might be required. One example here might be tabling of exp (x) for different values of x. One could use

$$e^x \doteq \sum_{p=0}^{n} \frac{x^p}{p!} \quad \text{but}$$

$$e^{-x} \doteq \sum_{p=0}^{n} \frac{(-x)^p}{p!} \quad \text{gives}$$

faster convergence, as well as better estimate of error.

On the whole, class and teacher learned much from the exercise. Subject to the model computer chosen, the student had a chance to be as creative as he could. It is easy to suggest problems at various levels of difficulty. A subroutine for square roots by approximation would be one. An improved model computer might contain trig functions in memory, and right and oblique triangles could then be solved. A lengthier problem would be a program for solution of a general cubic.

Summary

An advanced senior high mathematics class was introduced to the elements of the digital computer and its allied programming. The teacher educed from the class some necessary properties of operations of the computer, but a certain amount of didactic lecture seemed in order. Students then set to work on writing programs for the model computer, ranging from linear equations to systems of linear equations and quadratics. The week-long exercise in writing programs fostered clear and precise thinking on the solution of already familiar problems, and encouraged the use of maximum ingenuity to produce answers which were, in this case, useful digital programs. For more advanced students, increasingly sophisticated numerical problems can be devised.

Computer programming for high schools and junior colleges

DARREL G. LITTLEFIELD, *Alhambra Union High School, Martinez, California.*

A proposed course in computer programming for secondary schools.

THIS PAPER CONCERNS the recommendation of the adoption of a special course in digital computer programming at the high school or junior college level.

The recommended course would be rather selective, open only to those students who have a strong interest in and aptitude for mathematics and science.

The course could be made possible through the joint effort of high schools or junior colleges and highly interested groups, such as research laboratories, industrial organizations, or large business firms who possess the necessary digital computers.

It would be the responsibility of the high school or junior college to take steps to integrate the course in its system and the responsibility of the co-operative group to allow free released time on their computer so that the student programmers could actually make their own machine runs.

There will be problems to overcome in starting such a program, but their solutions can be realized through joint effort by school and allied groups. One of the most difficult will be the obtaining of personnel qualified to carry out the actual instruction. One solution would be long-range planning in the training of a mathematics or science teacher in computer programming. Another solution would be the provision of a trained person as instructor by the co-operative group.

The course would be organized in much the same way as any other science course, with regularly scheduled theory and laboratory sessions. The theory sessions would be held in the school plant, and the laboratory sessions would be scheduled for the installation where the computer is located.

As an alternative to the use of an industrial machine, a course could be introduced into a system where a student-designed-and-constructed computer is available. Although limited in scope, this type of course has been tried by some systems with noteworthy success. The author knows from his own experience that industrial firms are very willing to supply materials and technical advisors.

Both the growing national interest in and concern over the science curriculum of the secondary school and the ever increasing demand for qualified computer programmers seem to warrant such training at the high school and junior college levels.

The author received instruction in programming the IBM 650 through summer employment at the Lawrence Radiation Laboratory at Livermore, California, and realized that such instruction could be a real challenge to the student majoring in mathematics and science. He approached the Radiation Laboratory, IBM, and the high school district with the idea that a course in programming the IBM 650 be given to a selected group of seniors.

The pilot course, which was taught to a group of ten physics students during the second semester of the 1957–58 academic year, was mainly an extracurricular adventure, but it developed into a regularly scheduled credit course involving fifteen

small mathematics and small science majoring students with two instructors during the 1958–59 school year.

The trainees attended one two-hour theory session each week and one three-hour computer laboratory session each month. The majority of problems that were student programmed and machine run were taken from the mathematics and science courses in which the students were enrolled. One student, a junior boy, took on the very responsible task of programming the scheduling of students of Livermore High School which could result in near-optimum scheduling. A second student, a senior girl, was programming a problem related to the correlation between the results of the algebra placement tests and the success and failure of students in algebra.

As an outgrowth of this instruction, two students were able to design and construct a small binary computer which they demonstrated at the Western Joint Computer Conference held in San Francisco during March, 1959. Small units such as this one could readily be utilized as a teaching aid.

The computer course, as set up by the author with assistance from Mr. Herbert Thomas of the mathematics department, and the other programs mentioned before, have certainly proved that the high school student is able and needs only an opportunity to demonstrate his capabilities.

Course for computer programming for high schools and junior colleges

I. Name of course
   A. Computer programming
II. Student programmers
   A. High school
      1. Juniors
      2. Seniors
   B. Junior college
      1. Freshmen
      2. Sophomores
III. Objectives
   A. Basic understanding of the operation of the digital computer
   B. Appreciation of the power and application of the digital computer in modern technology and accounting
   C. Creation in the mind of the student of a more meaningful concept of number and equation
IV. Type of course and credit
   A. Elective course with academic credit
V. Prerequisites
   A. Strong interest in mathematics and science
   B. Good aptitude for mathematics and science
   C. Suitable background in mathematics and science
   D. Interest in computer programming
VI. Meeting times
   A. Classroom instruction—theory sessions
      1. Two to four two-hour sessions per week
   B. Machine run time—laboratory sessions
      1. One three-hour or two two-hour sessions per month
VII. Methods
   A. Visiting lecturers
      1. Representatives from industry, business, scientific laboratories
      2. Suggested topics for lectures
         a) History of computers
         b) Application
            (1) Past—solution of problems of tabulation
            (2) Present—solution of scientific and accounting problems
            (3) Future—solution of economic, social, and political problems
      3. Specific aspects of the programming process as seen by visiting lecturers
   B. Field trips—suggested tours
      1. Computer assembly plant

2. Data processing plants
   a) Business and industrial organizations
   b) Public service organizations
   c) Educational institutions
3. Allied manufacturing plants
   a) Calculator manufacturers
   b) Electronic equipment manufacturers

C. Teaching aids
1. Suggested materials
   a) Computer manual of operations
   b) Planning charts, coding forms
   c) Optimum programming chart
   d) Allied teaching publications, reference texts, computer journals
2. Films
   a) Available through computer manufacturers
3. Kits
   a) Manufactured
   b) Student produced
      (1) Training board
4. Charts, models
   a) Punch-card model
   b) Planning-chart model
   c) Internal-flow chart
5. Binary adder—training aid
   a) Logical add, subtract aid
   b) Student constructed aids
6. Desk calculator
   a) Hand, electric
7. Hard-card punch
   a) For placing instructions on input cards

VIII. Foundation in the mathematics
A. Binary number system
1. Compared with decimal system
   a) Conversion
      (1) To and from decimals
2. Addition—subtraction
   a) Addition table
3. Multiplication—division
   a) Multiplication table
4. Compared to octal system
   a) Facilitates conversion
      (1) Decimal to binary
      (2) Binary to decimal

B. Statistical—numerical analysis
1. Fundamentals of approximate measurement
2. Accuracy of measurement
3. Interpretation of observed measurement and recorded data

C. Boolean or logical algebra
1. Explain machine logic

IX. Principles of computation
A. Analysis
1. Problem analyzed

B. Theory of programming
1. Flow-chart block diagramming

C. Coding
1. Problem placed in machine language
2. Computation instructions
3. Coding techniques
4. Planning chart

D. Card punching
1. Instructions placed on input cards

E. Checking
1. Debugging—machine and hand

F. Machine run
1. To obtain calculated results

G. Evaluation—graphing

X. Machine organization and operation
A. Input—output
1. Card input—card output
2. Tape input
3. On-line printer output

B. Memory
1. Storage of word instructions
   a) Data information
   b) Instructions to facilitate arithmetic and testing operations

C. Arithmetic and testing operations
1. Addition—multiplication
2. Subtraction—division

3. Testing—branching

D. Control—checking
   1. Internal self-checking
   2. Console control
      a) Step-by-step observation of problem
      b) Control and guidance of the computer in the solution of problem
      c) Detection of errors
      d) Trouble analysis—debugging

Conclusion

The description of the course in computer programming proposed for high schools and junior colleges is based primarily on the type of course that was taught at Livermore Union High School, Livermore, California, during the 1958–59 school year and which is being introduced into the Alhambra Union High School, Martinez, California, this year.

The basic objective in this course is not to train students as computer technicians but rather to expose them to an avenue of endeavor which will tend to enlighten and enrich. As Mr. Aaron Buchman, a high school mathematics instructor in Buffalo, New York, stated, "Computer programming is a step in the direction of the fuller utilization of the scientific and mathematical abilities of young people."

The statement of prerequisites should not be interpreted as meaning that only the very brightest students are qualified for such training. There is a good opportunity for students of varying abilities in the field of programming. In the course taught at Livermore, student trainees of only average ability were able to complete the course successfully. The desire to succeed is the most important qualification.

In programming there is a certain amount of just plain mechanical or pencil work called coding which applies to the placing of the problem in machine language form. Coding is the simplest part of the programming process and is less demanding on the programmer than the problem analysis, debugging, machine running, and the interpretation of computed results. The course as it is taught in high school would normally be geared to the coding process, but some students are able to grasp the very fundamental concepts of programming, and they could be considered as student programmers.

The study of the binary system, Boolean algebra, and elementary numerical analysis is not a necessary part of such instruction at the high school level, but with an understanding of these mathematical concepts, a more realistic concept of programming will be realized by the student.

The author knows from his own experience and from having read of the experiences of others that such a course is most feasible and could be introduced into a large number of high schools and junior colleges. There are hundreds of IBM 650 computers in use in the United States and probably just as many research laboratories, industrial concerns, and business organizations who are willing to co-operate with a comparable number of high schools or junior colleges in starting courses.

The proposal for a course in computer programming is not confined to this machine alone. Other computers built by IBM, Remington Rand, Burroughs, and others, could be programmed by high school and junior college students.

The author strongly recommends that high schools and junior colleges look to the computer as a natural educational resource. Like other school resources, the computer can be employed to foster positive and rewarding scholastic growth for the youth entrusted to the school's care.

Bibliography

Buchman, Aaron L. "Computer Programming and Coding at the High School Level," *Journal for the Eleventh Annual Meeting of Association for Computing Machinery*, University of California at Los Angeles (August, 1956).

"Computer Programming by Students of Livermore Union High School," *Journal for the Western Joint Computer Conference*, San Francisco (March, 1959).

Tillitt, Harley. "Computer Programming for Young People," *Journal of the Association for Computing Machinery* (October, 1958).

Digital computer—nonelectronic

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Some interesting computations on one of our oldest digital computers.

NOW THAT WE ARE well embarked on the computer age, it might be profitable to call the pupil's attention to the built-in computer from which he is inseparable. This computer consists of a set of counters (ten fingers) and some very elementary techniques. The techniques required here are very elementary indeed, consisting of the ability to multiply simple numbers, to count in multiples of ten, and to perform simple addition.

The mechanics of the operation consist of arranging the fingers in certain combinations. The size of the multiplicand and multiplier determines the value given to counters. If we wish to multiply a number ranging from 6 to 10 by another number within this same range, number the fingers of each hand, starting with the thumb, 6, 7, 8, 9, 10, respectively. To multiply, close the fingers of one hand up to and including the finger representing the multiplicand and, keeping them together, point them generally downward, allowing the loose fingers to point upward. We shall hereafter refer to the down or closed fingers as the primary fingers, and the up or loose fingers as the secondary fingers. Adjust the right hand in like manner for the multiplier and bring the hands together. To count, allow 10 for each closed or primary finger of both hands. Add to this the product of the number of loose secondary fingers on the left hand by the number of loose secondary fingers on the right. The result is the product desired.

To multiply 8×7 for example, the thumb and first two fingers of the left hand (8) will be placed next to the thumb and first finger of the right (7) as in Figure 1. The partial product is 10 for each finger in the closed primary group (50). To this is added the product of the number of loose or secondary fingers on the left hand by the number of loose or secondary fingers on the right (2×3=6). The sum is 56, the product of 8×7.

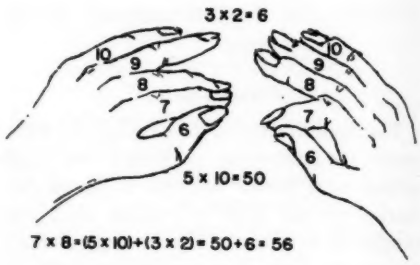

Figure 1

The same method may be used to multiply a number in the range 16 to 20 by another number ranging from 16 to 20. Fingers are numbered 16, 17, 18, 19, 20, starting with the thumb. In counting, start with an additive constant of 200, allow 20 each for the primary fingers, and add the product obtained as before from the secondary fingers of the right and left hands respectively. For example, to multiply 19×19 using this system:

Additive constant	=200
8 primary fingers at 20 each	=160
Secondary fingers 1×1	= 1
Total	=361

19×19=361

To multiply a number ranging from 11 to 15 by another in this same range, assign

values from 11 to 15 to the fingers in the same manner as previously. Close the fingers of the left hand up to and including the finger representing the multiplicand, and leave the rest loose, pointing upward. These are the primary and secondary fingers respectively. Arrange the right hand likewise for the multiplier. Start with an additive constant of 100, and count 10 for each counter. For the additional partial product, use the primary fingers this time (the same being used as counters), the number of fingers on the left hand being multiplied by the number of fingers on the right. Adding these three numbers will complete the desired product. As an example, to multiply 14×13:

Additive constant	$=100$
7 primary fingers at 10 each	$=70$
Primary fingers 4×3	$=12$
Total	$=182$

$14 \times 13 = 182$

To multiply a number ranging from 21 to 25 by another in this same range, number the fingers appropriately, start with an additive constant of 400, allow 20 for each primary finger, and add the product obtained from the number of primary fingers on the left and right hands respectively.

Further information for multiplying two integers, each within the same half-decade of the number system, is supplied in Table 1. The integers are grouped in decades, each decade starting from one in one's place and including the multiple of ten directly above. For example: 31 to 40 is decade number three, forty being included in the three decade. It should be further noted that the numbers 1 to 10 comprise what is referred to in this treatment as the zero decade.

The lower and upper five digits of each decade are keyed into the table as *L* and *U* respectively. This is done because these two half-decades require different treatment. It will be noted from the table that when two numbers, each from the same lower half-decade are to be multiplied, the partial product is obtained from the primary fingers, while if numbers from an upper half-decade are to be multiplied, the secondary fingers are used. Otherwise stated, the use of primary and secondary fingers for partial products alternates every half-decade.

The product of any two numbers lying within the same half-decade may be obtained from the table by finding the correct constant *K*, multiplying the number of counters by the counter value given, using the correct fingers, primary or secondary, for the partial product, and add-

TABLE 1

Decade D	Half-decade	Additive constant K	Value of counters C	Fingers used to produce partial products
9	U	9000	100	S
	L	8100	90	P
8	U	7200	90	S
	L	6400	80	P
7	U	5600	80	S
	L	4900	70	P
6	U	4200	70	S
	L	3600	60	P
5	U	3000	60	S
	L	2500	50	P
4	U	2000	50	S
	L	1600	40	P
3	U	1200	40	S
	L	900	30	P
2	U	600	30	S
	L	400	20	P
1	U	200	20	S
	L	100	10	P
0	U	—	10	S
	L	—	—	P

Definitions

K = additive constant
C = counter value
D = the value of the decade
L = lower half of decade (Example: 21 to 25)
U = upper half of decade (Example: 36 to 40)
P = primary fingers "down" or "closed"
S = secondary fingers "up" or "loose"

ing the three results together. For example, the solution of the problem 81×83 appears thus:

Solution:

K	$=6400$
4 primary fingers at 80 each	$=320$
Primary fingers 1×3	$=3$
Total	$=6723$

$81 \times 83 = 6723$

For further simplification and use of this system without constant reference to the table, several formulas are useful. Where a product is desired whose factors fall within some specific half-decade, the constant K and the counter C may be obtained by the use of formulas. Using D for decade value, the formulas for the lower half-decades are:

$$K_L = 100D^2, \qquad C_L = 10D.$$

If upper half-decade factors are used, the formulas are:

$$K_U = 100D(D+1), \qquad C_U = 10(D+1).$$

Heretofore we have been concerned only with an organized counting of the fingers to achieve the multiplication process. An explanation as to why the system produces the correct answer is most certainly apropos. To begin with, the two number factors may be thought of as two binomials. In multiplying 13×14 as binomials for example, we have $13 \times 14 = (10+3)(10+4) = 100 + 70 + 12 = 182$. In our system of multiplication, the system currently being presented, we start with $K = 100$, and add 70 (7 primary fingers at 10 each) and add 3×4 (the respective primary fingers of the two hands). To take another example, $22 \times 24 = (20+2)(20+4) = 400 + 120 + 8 = 528$. In our system, we start with $K = 400$, add 120 (6 primary fingers at 20 each), and add 8 (primary fingers, 2×4). These two examples selected from lower half-decade brackets parallel the algebraic product of two binomials exactly!

In the upper half-decade brackets, the explanation takes a slightly different turn. In our system, 17×19 becomes $200 + 120 + 3 = 323$. Using binomials, $17 \times 19 = (10 + 7)(10 + 9) = 100 + 160 + 63 = 323$. There seems to be no similarity, but try again with $17 \times 19 = (20 - 3)(20 - 1) = 400 - 80 + 3 = 323$. Now use the fingers again and count another way. If the secondary fingers are used as counters for subtracting instead of the primary fingers for adding, we can start with 400, subtract 80 (4 secondary fingers at 20 each) and add 3. As another example, 8×9 becomes $(10-2)(10-1)$. If we use secondary fingers for counters for subtraction, and start with the knowledge that we are working downward from 100, we count 3 secondary fingers for 30 to be subtracted, and have a partial product of 2 to be added, giving us $100 - 30 + 2 = 72$. Again we parallel binomial multiplication exactly!

Previously in this presentation, only pairs of factors each within the same half-decade number bracket have been used. For example, using Table 1, a factor from the range 16 to 20 can only be multiplied by a factor from the range 16 to 20. We now propose to expand the system and its usefulness by removing this limitation. The expanded system is summarized in Table 2.

The main reason for different treatment of the multiplication process in the case of products whose factors fall within different half-decade brackets is that in this situation, as contrasted with the previously presented situation, the left-hand counters must be assigned different number values than the right-hand counters. As before, the primary fingers are used as counters. In Table 2, the counter values assigned to the left-hand primary fingers have been designated C_L; for the right hand, C_R.

Some examples should clarify the uses that can be made of Table 2. If we wish to multiply 22×34, this constitutes a number from the second decade (2), lower half (L), times a number from the third decade (3), lower half (L). This appears in the table in the category $2L \times 3L$. The number of primary fingers are: left hand—2, right

TABLE 2

Factors				K Value	Counter value		P or S
1		2					
D	H	D	H		C_L*	C_R*	
0	U	1	L	50	10	5	P
0	U	2	L	100	20	5	P
0	U	3	L	150	30	5	P
0	U	4	L	200	40	5	P
0	U	5	L	250	50	5	P
0	U	1	U	50	20	10	S
0	U	2	U	100	30	10	S
0	U	3	U	150	40	10	S
0	U	4	U	200	50	10	S
0	U	5	U	250	60	10	S
1	L	1	U	150	15	10	P
1	L	2	U	250	25	10	P
1	L	3	U	350	35	10	P
1	L	4	U	450	45	10	P
1	L	5	U	550	55	10	P
1	L	2	L	175	25	15	S
1	L	3	L	275	35	15	S
1	L	4	L	375	45	15	S
1	L	5	L	475	55	15	S
1	U	2	L	300	20	15	P
1	U	3	L	450	30	15	P
1	U	4	L	600	40	15	P
1	U	5	L	750	50	15	P
1	U	2	U	350	30	20	S
1	U	3	U	500	40	20	S
1	U	4	U	650	50	20	S
1	U	5	U	800	60	20	S
2	L	2	U	500	25	20	P
2	L	3	U	700	35	20	P
2	L	4	U	900	45	20	P
2	L	5	U	1100	55	20	P
2	L	3	L	575	35	25	S
2	L	4	L	775	45	25	S
2	L	5	L	975	55	25	S
2	U	3	L	750	30	25	P
2	U	4	L	1000	40	25	P
2	U	5	L	1250	50	25	P
2	U	3	U	850	40	30	S
2	U	4	U	1100	50	30	S
2	U	5	U	1350	60	30	S
3	L	3	U	1050	35	30	P
3	L	4	U	1350	45	30	P
3	L	5	U	1650	55	30	P
3	L	4	L	1175	45	35	S
3	L	5	L	1475	55	35	S
3	U	4	L	1400	40	35	P
3	U	5	L	1750	50	35	P
3	U	4	U	1550	50	40	S
3	U	5	U	1900	60	40	S
4	L	4	U	1800	45	40	P
4	L	5	U	2200	55	40	P
4	L	5	L	1975	55	45	S
4	L	6	L	2375	65	45	S†
4	U	5	L	2250	50	45	P
4	U	6	L	2700	60	45	P
4	U	5	U	2450	60	50	S
4	U	6	U	2900	70	50	S
5	L	5	U	2750	55	50	P
5	L	6	U	3250	65	50	P

Explanation of column headings

Factor 1—multiplicand
Factor 2—multiplier
D—decade of the given factor
H—half-decade bracket of the factor

C_L—counter value, left hand*
C_R—counter value, right hand*
P or S—primary or secondary fingers used to obtain partial product

* Primary fingers used as counters

† It was originally intended to limit the table to the five decade. This was done until an exception seemed indicated. Certain differences in K and C_L values appearing in the table seemed to have value enough to show in all cases. In the last stages of the table, the six decade was included when these differences would not otherwise be shown.

hand—4. We use the secondary fingers to obtain the partial product. From the table:

K	$=575$
$C_L=35$; 2×35	$=70$
$C_R=25$; 4×25	$=100$
$S=$secondary; $3\times1=$	3
Total	$=748$

$22\times34=748$

For another illustration, to multiply 16×42, we are in the category $1U\times4L$. This gives primary fingers: left hand—1, right hand—2. We use primary fingers for the partial product. From the table:

K	$=600$
$C_L=40$; 1×40	$=40$
$C_R=15$; 2×15	$=30$
$P=$primary; $1\times2=$	2
Total	$=672$

$16\times42=672$

It may appear that all has now been said in regard to the combinations of half-decade number brackets; that, for example, no other combinations of K, C, and primary and secondary choices can be used to achieve the multiplication process.

As a check, we might try multiplying 14×13 as was done in one of these earlier paragraphs, but with the difference that we now wish to use secondary fingers for the partial product. (Formerly we used primary fingers, $K=100$, $C=10$.) Let us now use $K=75$ and $C=15$. The problem appears thus:

K	$=75$
7 primary fingers at 15	$=105$
Secondary fingers $1\times2=$	2
Total	$=182$

$14\times13=182$

These same values of K and C may be tried with the other combinations of numbers within this same half-decade number complex. They will be found to satisfy the conditions of the problem, providing the correct solutions.

It is felt that the scope of both tables is sufficient to allow the reader an inkling of the relationships and interplay of the numbers as applied to this treatment. It is hoped that, aside from the purely mechanical aspects of this system, sufficiently provocative material has been here provided to stimulate further investigations of our number system.

What's new?

BOOKS

HIGH SCHOOL

Solid Geometry, J. L. Simpson. New York: Harper and Brothers, 1960. Cloth, viii+97 pp. $2.75

COLLEGE

Elements of Calculus, (2nd ed.), Thurman S. Peterson. New York: Harper and Brothers, 1960. Cloth, xii+519 pp. $6.50

Calculus with Analytic Geometry, Thurman S. Peterson. New York: Harper and Brothers, 1960. Cloth xii+586 pp. $7.50

MISCELLANEOUS

The Foundations of Arithmetic, Gottlob Frege, translated by J. L. Austin. New York: Harper and Brothers, 1960. Paper, 119 pp., $1.25.

The Intelligent Man's Guide to Science, Vol. 1, *The Physical Sciences;* Vol. 2, *The Biological Sciences*, Isaac Asimov. New York: Basic Books, Inc., 1960. Cloth, Vol. 1, xiv+382 pp.; Vol. 2, vii+383 to 853 pp., $15.00 per set.

Soviet Education Programs, William K. Medlin, Clarence B. Lindquist, and Marshall L. Schmitt. U.S. Department of Health, Education, and Welfare, Office of Education, Bulletin 1960, No. 17. Washington, D.C.: U.S. Government Printing Office, 1960. Paper, xvii+281 pp. $1.25

Complex numbers and loci

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Graphing in the complex plane is a technique for improving background in complex numbers as well as for review of important algebraic processes.

THE FACT THAT THE GRAPH of the locus of (1) $|z-2i| = |3z+1|$, (2) $|z+i|+|z-2i| = 6$, and (3) $|z-3| > |z+3i|$ in the complex plane, is a circle, an ellipse, and a half-plane respectively has interested students in second-year algebra.

When college preparatory students are grouped according to ability in mathematics, it becomes necessary to enrich the curriculum for the upper-level groups. For the upper-level grouping in second-year algebra, it has been found that the usual coverage of complex numbers might be supplemented by a unit on the graphing of loci which result from the absolute value of complex numbers appearing in equations. The writer has prepared such a unit and has taught it to an upper-level group in second-year algebra in Wauwatosa Senior High School.

The nature of this material requires that the student shall have had the usual second-year-algebra curriculum on complex numbers and also a unit on conic sections graphed in the Cartesian plane.

The unit on complex numbers and loci was prepared because it is believed that the material provides an opportunity for the student to review and to get a deeper understanding of many concepts and processes studied in his previous work in mathematics. These concepts and processes include: (1) complex numbers and the complex plane, (2) conic sections, (3) graphs of straight line, circle, parabola, ellipse, hyperbola, equilateral hyperbola, portions of the plane (graphs of inequalities), (4) completing the square, (5) algebraic loci, (6) operations on inequalities and the solution of inequalities, (7) absolute value, (8) radical equations, (9) squaring binomials, and (10) factoring.

The unit on complex numbers and loci was developed in the following manner.

Complex numbers are numbers of form

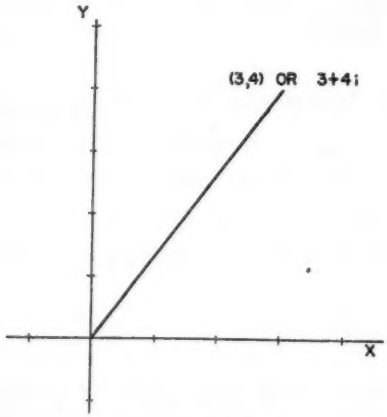
$$x+iy$$

where x and y are real numbers. z will be used to denote a complex number, thus

$$z=x+iy.$$

When writing complex numbers in this way, it can be shown that there exists a one-to-one correspondence between complex numbers and ordered pairs of real numbers (x, y). Since the Cartesian plane is made up of ordered pairs of real numbers, each complex number can be uniquely represented graphically by a point in the Cartesian plane. Thus the complex number $3+4i$ is uniquely represented in the Cartesian plane by the point (3, 4), as seen in Figure 1, page 230. When the Cartesian plane is used in this way, it is referred to as the complex plane or z-plane. The figure in which complex numbers are plotted is called the Argand diagram.

In the number $z=x+iy$, when $y=0$, then $z=x+i\cdot 0=x+0=x$, which is a real number. The only place in the complex plane where $y=0$ is the x-axis. It can be

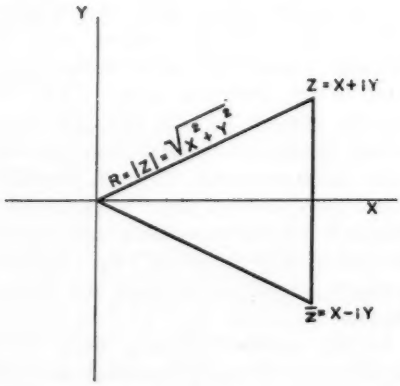


Complex plane
Figure 1

said that all points on the x-axis represent real numbers and that all real numbers can be represented as points on the x-axis.

In a similar way, in the complex number $z=x+iy$, when $x=0$ and $y\neq 0$, $z=0+iy=iy$, which is a pure imaginary number. The only place in the complex plane where $x=0$ is the y-axis. It can be said that all points on the y-axis except (0, 0) represent pure imaginary numbers and that all pure imaginary numbers can be represented as points on the y-axis.

The rest of the complex plane is made up of complex numbers $z=x+iy$ where $x\neq 0$ and $y\neq 0$, shown in Figure 2.



Complex plane
Figure 2

When $z=x+iy$, we write

$x=Re(z)=$ real part of z.
$y=Im(z)=$ imaginary part of z.
$r=|z|=$ absolute value of z, sometimes called the modulus of z.
$x-iy=\bar{z}=$ conjugate of z.

By the Pythagorean theorem, the $|z|$, i.e., the length of the vector from the origin to point z, is equal to the square root of the sum of the squares of the real and imaginary components of z. We write

$$|z|=\sqrt{x^2+y^2}.$$

This algebraic expression for the absolute value of z enables us to graph the loci of algebraic equations involving the absolute value of complex numbers. For example, if $|z|=1$, then we have

$$\sqrt{x^2+y^2}=1 \qquad \text{by substitution}$$

and

$$x^2+y^2=1 \qquad \text{by squaring.}$$

Therefore the set of points in the complex plane with co-ordinates of x and y, which satisfies the condition of the equation $|z|=1$, is a circle with the center at the origin and a radius of 1, as in Figure 3.

In a similar way, loci of other equations and inequalities involving complex numbers, such as those given in the first para-

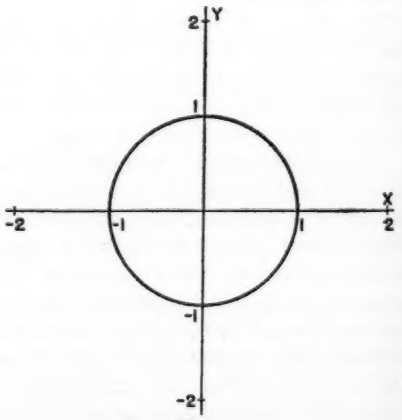


Figure 3

graph of this paper, can be determined and graphed. Following are the detailed solutions of the two equations and the inequality appearing in the first paragraph, plus other examples with the detailed parts of their solutions omitted.

Example 1.

$$|z-2i| = |3z+1|$$
$$|x+iy-2i| = |3x+3iy+1|$$
$$|x+i(y-2)| = |(3x+1)+3iy|$$
$$\sqrt{x^2+(y-2)^2} = \sqrt{(3x+1)^2+9y^2}$$
$$x^2+(y-2)^2 = (3x+1)^2+9y^2$$
$$x^2+y^2-4y+4 = 9x^2+6x+1+9y^2$$
$$8x^2+6x+8y^2+4y = 3$$
$$8\left(x^2+\frac{6}{8}x\right)+8\left(y^2+\frac{4}{8}y\right) = 3$$
$$8\left(x^2+\frac{3}{4}x+\frac{9}{64}\right)+8\left(y^2+\frac{1}{2}y+\frac{1}{16}\right) =$$
$$3+8\cdot\frac{9}{64}+8\cdot\frac{1}{16}$$
$$8\left(x+\frac{3}{8}\right)^2+8\left(y+\frac{1}{4}\right)^2 = \frac{37}{8}$$
$$\left(x+\frac{3}{8}\right)^2+\left(y+\frac{1}{4}\right)^2 = \frac{37}{64}.$$

The set of points in the complex plane which satisfies the condition

$$|z-2i| = |3z+1|$$

is a circle with radius of $\sqrt{37}/8$ and center at the point $(-\frac{3}{8}, -\frac{1}{4})$.

Example 2.

$$|z+i|+|z-2i| = 6$$
$$|x+iy+i| = 6-|x+iy-2i|$$
$$|x+i(y+1)| = 6-|x+i(y-2)|$$
$$\sqrt{x^2+(y+1)^2} = 6-\sqrt{x^2+(y-2)^2}$$
$$x^2+y^2+2y+1 =$$
$$36-12\sqrt{x^2+(y-2)^2}+x^2+y^2-4y+4$$
$$6y-39 = -12\sqrt{x^2+(y-2)^2}$$
$$2y-13 = -4\sqrt{x^2+y^2-4y+4}$$
$$4y^2-52y+169 = 16(x^2+y^2-4y+4)$$
$$4y^2-52y+169 = 16x^2+16y^2-64y+64$$
$$16x^2+12y^2-12y = 105$$
$$16x^2+12(y^2-y) = 105$$
$$16x^2+12\left(y^2-y+\frac{1}{4}\right) = 105+12\cdot\frac{1}{4}$$
$$16x^2+12\left(y-\frac{1}{2}\right)^2 = 108$$
$$\frac{16x^2}{108}+\frac{12}{108}\left(y-\frac{1}{2}\right)^2 = \frac{108}{108}$$
$$\frac{4}{27}x^2+\frac{1}{9}\left(y-\frac{1}{2}\right)^2 = 1$$
$$\frac{x^2}{\frac{27}{4}}+\frac{\left(y-\frac{1}{2}\right)^2}{9} = 1.$$

The set of points in the complex plane which satisfies the condition

$$|z+i|+|z-2i| = 6$$

is the ellipse, graphed in Figure 4,

$$\frac{x^2}{\frac{27}{4}}+\frac{(y-\frac{1}{2})^2}{9} = 1.$$

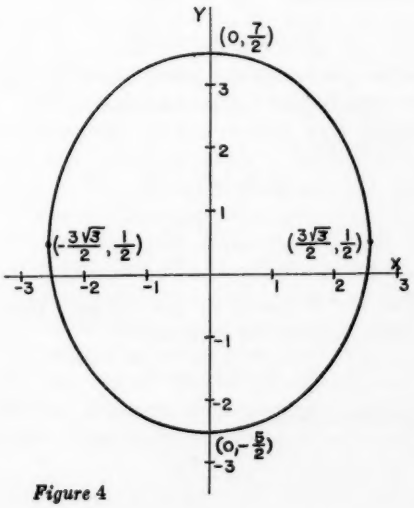


Figure 4

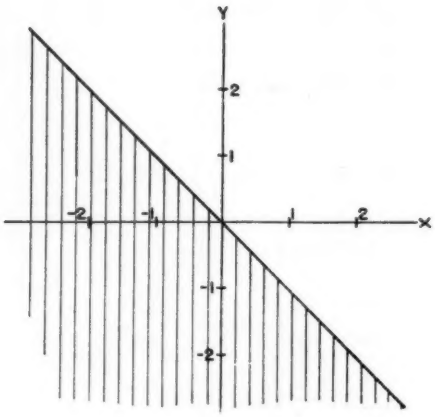


Figure 5

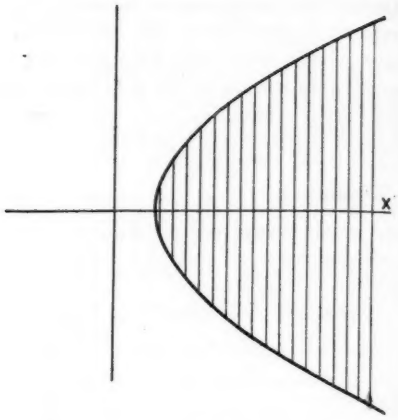


Figure 6

Example 3.

$$|z-3| > |z+3i|$$
$$|x+iy-3| > |x+iy+3i|$$
$$|x-3+iy| > |x+i(y+3)|$$
$$\sqrt{(x-3)^2+y^2} > \sqrt{x^2+(y+3)^2}$$
$$(x-3)^2+y^2 > x^2+(y+3)^2$$
$$x^2-6x+9+y^2 > x^2+y^2+6y+9$$
$$-6x > 6y$$
$$x < -y.$$

The set of points in the complex plane which satisfies the condition

$$|z-3| > |z+3i|$$

is the portion of the plane below the line $x=-y$. Figure 5 illustrates this.

Example 4.

$$Re(z) \geq |z-2|$$
$$x \geq \frac{4+y^2}{4}.$$

The set of points in the complex plane which satisfies the condition

$$Re(z) \geq |z-2|$$

is the parabola

$$x = \frac{4+y^2}{4}$$

as graphed, and the portion of the plane inside the parabola, as shown in Figure 6.

Example 5.

$$|z| < 4$$
$$x^2+y^2 < 16.$$

The set of points in the complex plane which satisfies the condition $|z| < 4$ is the portion of the plane enclosed by a circle with radius of 4 and center at the origin. The points on the circle are not in the graph. This can be seen in Figure 7.

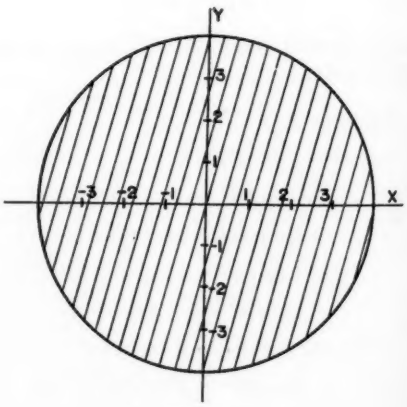


Figure 7

Example 6.

$$|z-4|-|z|=2$$

$$\frac{(x-2)^2}{1}-\frac{y^2}{3}=1.$$

The set of points in the complex plane which satisfies the condition

$$|z-4|-|z|=2$$

is the hyperbola

$$\frac{(x-2)^2}{1}-\frac{y^2}{3}=1$$

as graphed in Figure 8.

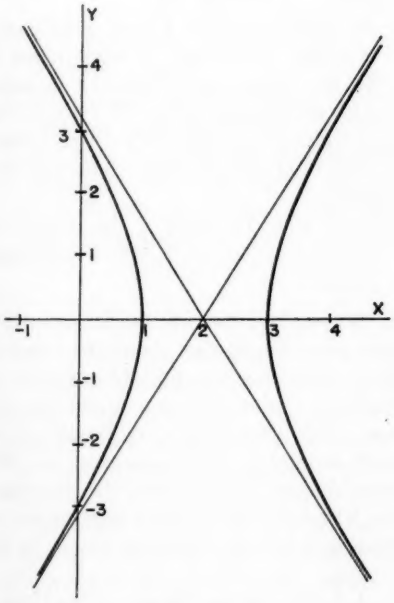


Figure 8

Example 7.

$$1<|z+3-2i|<3$$

$$1<(x+3)^2+(y-2)^2<9.$$

The set of points in the complex plane which satisfies the condition

$$1<|z+3-2i|<3$$

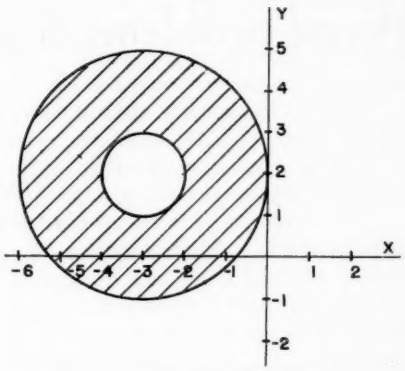


Figure 9

is the portion of the plane included between two concentric circles with radii of 1 and 3, and centers at the point $(-3, +2)$. This plane is shown in Figure 9.

Outcomes

In a test on this material, the question "What do you feel you have gained from the study of this material?" was asked of the students. It was interesting to observe that all the afore-mentioned objectives in preparing this material for the students appeared in their responses to this question. In addition, the idea that it was interesting to watch the developments in a solution of an equation of this type appeared a number of times in the responses made by the students.

The material described in this paper is recommended for the teacher who wishes to enrich the curriculum in second-year algebra. The writer has observed that the study of this material on complex numbers and loci not only gives the student a better understanding of complex numbers, but also gives him a review and a deeper understanding of conic sections, various kinds of graphs, inequalities, algebraic loci, absolute value, radical equations, and algebraic processes such as completing the square, squaring binomials, and factoring.

Word problems or problems with words

JEROME MANHEIM, *Montclair State College, Upper Montclair, New Jersey.*

This writer states, "With the current emphasis on logical analysis and the study of structure the role of the 'word problem' should be re-evaluated."

THE METAMORPHOSIS that "word problems" have undergone in our lifetime has resulted from psychological investigations into motivation. Evidence in support of the change comes from researchers and teachers, whenever they have asked about the "here and now" as opposed to the "there and then." Despite this major breakthrough, the "word problem" remains a generator of fear and frustration for many students. This article will be concerned with three aspects of elementary "word problems." If we cannot make them our welcome friends, it is hoped to remove at least some of the apprehensions traditionally associated with their appearance.

REAL AND IMAGINARY WORD PROBLEMS

We are accustomed to thinking of a "word problem" as "a problem with words" (excepting, of course, simple directions, such as "Solve for x," or "Find the roots of the following equations."). If we ask our students, or ourselves, why such a problem is more difficult than a non-word problem, we are apt to find the difficulty attributed to "the nonmathematical nature of the problem." Ineptitude is frequently excused with the reminder that the solver is not, after all, supposed to be a physicist, social scientist, etc. How good or how poor this rationale is will emerge later. For the present we will distinguish between "real word problems" and "imaginary word problems."

A "real word problem" is one which employs English or other nonmathematical language to ask a question about a physical (chemical, economical, mathematical, biological, social, etc.) phenomenon, the laws governing the particular phenomenon being presupposed. The following are illustrative problems.

1. A train going east was overtaken in six hours by one leaving the same place two hours later and traveling sixteen miles per hour faster. Find the rate of each.[1]
2. An inductance of 1 henry, a resistance of 3,000 ohms, and a capacitance of 5×10^{-7} farad are connected in series with an e.m.f. of $100e^{-1000t}$ volts. If the charge and current are both zero when $t=0$, find the current when $t=0.001$ sec.[2]
3. Show that the sum of the roots of a quadratic equation with real coefficients is real.

To solve each of these problems we need to have prior knowledge about the relationships governing quantifiable words. In Problem 1 the relevant quantities are time, distance, and rate; in Problem 2 we must ultimately have access to the differential equation that describes the situation where the three circuit parameters are connected in series with an e.m.f.; in the

[1] J. C. Stone and V. J. Mallory, *A First Course in Algebra*, New Edition, (Chicago: Benj. H. Sanborn & Co., 1936), p. 228, Problem 30.

[2] H. W. Reddick and Frederick H. Miller, *Advanced Mathematics for Engineers*, 3rd ed., (New York: John Wiley & Sons, Inc., 1955), p. 90, Problem 40.

last problem our prior knowledge is the quadratic formula. In each case, something external must be known or determined before the problem can be solved.

An "imaginary word problem" differs from a "real word problem" in only one respect, the governing laws, those which relate the quantifiable words, are explicitly enunciated. Here are some examples of "imaginary word problems":

A. The formula for changing degrees on a Fahrenheit thermometer to degrees on a centigrade thermometer is

$$C=5/9(F-32).$$

If $F=-22$, find the value of C.[3]

B. A resistance coil has total resistance R. The coil is to be divided into two coils, which are then to be connected in parallel. If the resistance R_e, equivalent to two parallel resistances R_1 and R_2, is given by

$$\frac{1}{R_e}=\frac{1}{R_1}+\frac{1}{R_2},$$

how should the coil be divided to make R_e a maximum?[4]

C. The fundamental trantort unit on the western hemisphere of the planet Jaros is the fratezat; on the eastern hemisphere it is the caliope. The fratezat, F, and the caliope, C, are related by the formula:

$$C=5/9(F-32).$$

If $F=-22$, how much is C?

A characteristic of these problems, as the reader has probably observed by comparing Problem A and Problem C, is that their solution is entirely independent of word interpretation and, hence, of physical meaning. No excuse about lack of training in the related discipline is valid in these cases. It is important for the student to understand this and then to exercise (for the particular problem) his prerogative to invent. After some encouraging, a student decided (Problem B) that a resistance coil looked like a square. When it was divided into two parts and connected in parallel, it looked like a corporal's stripes. Actually, these symbols are just as satisfactory as the accepted ones; neither looks anything like a resistance coil, nor is there any reason that they should.

We propose, then, to dispose of the "imaginary word problem" by deciding that it is not really a word problem after all. If the nature of such problems is explained to students, if they are taught to look for and focus on the intrinsic nature of the problem rather than the surrounding verbiage, most of the difficulty accompanying this type of problem should disappear.

It is true that the dichotomy is not so clean as we might wish. If we are willing to call the quadratic formula an external relationship (Problem 3), then what about the laws of arithmetic and algebra necessary to the solution of all the problems? These are not explicitly stated, and yet they are necessary. A practical means of resolving this semantic difficulty is to differentiate between language, axioms, and methodology on the one hand and derived results on the other. If a student does not know the language and rules of algebra, he probably should not have been exposed to the problem in the first place; if he does, our distinction is unambiguous.

The technical mechanism of solving word problems

If we have a "real word problem," then it is given entirely in the nonmathematical language (e.g., Problems 1 and 3 on page 234), or it is easily transcribed into this language (e.g., Problem 2). The laws relevant to the solution can be similarly expressed. It is, however, the transcription from the nonmathematical tongue to the mathematical one that we seek, and since this is generally the more difficult, it is reasonable to inquire into the reason for the choice.

[3] J. C. Stone, *op. cit.*, p. 240, Problem 14.
[4] J. R. Britton, *Calculus* (New York: Rinehart & Co., Inc., 1956), p. 89, Problem 2.

The reason we transform the expression of the problem from the nonmathematical tongue (e.g., English) to the mathematical tongue is that we have greater manipulative facility in the latter area. In fact, within mathematics itself we often effect transformations to facilitate solution. A problem that is presented in one coordinate system may be transformed to another. If the problem is topological in character, areas may be transformed into points, etc., as in the classical solution to the Königsberg Bridge problem. In differential equations we introduce operators because they assume an algebraic role, and we are more at home with algebra than with differential equations.

After the problem is solved in the transform language, we translate back into the original tongue to exhibit the answer.

Viewed another way, what is involved is a double mapping. The problem originally given in one space (e.g., English) is mapped into another space (mathematics). Transformations within this latter space are made according to established rules. At the proper point the inverse mapping, from the mathematics to the English (or the x' to the x, or the algebra to the solution of the differential equation), is effected.

This tells us why we use mathematics to solve word problems; it doesn't tell us how to express the problem mathematically.

Obtaining the solution

If all goes well, then the chronology for solving a word problem will read:

1. The problem is understood.
2. The laws relevant to the situation are selected.
3. The general laws in statement 2 and the corresponding "words" in statement 1 are made to interact in order to get mathematical expressions that are mappings of the word data.
4. The mathematical symbols are manipulated according to an accepted methodology.
5. The manipulation ceases when transformation of the solution is obtained.
6. The inverse mapping occurs, and the answer is exhibited.

A number of things may happen which can disrupt or delay this program. Numerous writers have discussed problem solving in general. We shall restrict our examination to some difficulties that arise specifically from the word nature of the problem.

Choosing the laws relevant to the situation

For definiteness of approach we will focus on the train problem on page 234, though the discussion has general applicability. We really have two problems here: how do we select a law for consideration, and how do we know when to cease looking for more laws?

As we read over the problem, those words that are amenable to quantification are selected; these are clearly distance, time, and rate. (We may have some ambivalent feelings about the word "east" because this refers to direction, a concern that is frequently relevant to a problem. This shall be kept in mind.) We now cast about, among our store of preinformation, for laws relating some or all of the quantifiable words. Either we know that "distance = (rate)(time)," or we don't. If the formula is entirely foreign to us, i.e., it is not part of our preinformation and is not derivable from known data, then we will not solve the problem, and the excuses based upon the nonmathematical character of the problem are clearly valid here. If, on the other hand, we know the formula, then we have forfeited all rights to this excuse, as we shall see later. Assuming that we do know the formula, we lay it out (write it down) for consideration. This answers the first part of our question.

We must next decide when to stop looking for formulas relating the quantifiable words. There is no complete set of universal rules here, but there are useful suggestions. In the first place, we want enough

laws exhibited so that all the relevant quantities are mentioned at least once. We have a law relating distance, time, and rate; at this point, we must make up our mind about "east." Although it is possible to develop a metarule that will tell us about the necessity of including a rule about a direction, it will be of limited applicability; it will not resolve corresponding difficulties that arise in other problems, where "hotter," "richer," "heavier" occupy the questionable role. Nonetheless, the decision must be made. Assuming the absence of a semantic problem, the decision about the relevance of a word is essentially a mathematical one. "How does the word 'east' affect the structure of the problem?" Once we have decided that "east" is a mathematical dummy, we will be in a position where all the relevant words in the problem that are concerned with magnitude appear in the chosen set of law(s)—in this case, "$D=rt$." Although this fact does not, of course, guarantee that we need not seek additional governing laws that are external to the problem, it suggests that we *may* have sufficient law(s) for the problem. Barring prior familiarity with the problem type, we won't know whether or not we need additional relationships until we have proceeded further with the problem—either solving it, in which case it is clear that we had employed sufficient law(s), or failing of a solution in a manner indicating that we need additional relationships.

Generally speaking, we should proceed with the solution as soon as we have sufficient laws to relate all the relevant words. For, in most elementary problems where not enough laws have been used, the evidence for this will appear early in the attempted solution in the guise of a deficient set of equations. The alternative choice of seeking additional relationships can be dangerous, especially for the novice. The main danger is that laws not independent of the preceding ones will be chosen. For example, in the present problem the solver might propose

$$\text{rate}=\frac{\text{distance}}{\text{time}}$$

as a new relationship. In the absence of error, he is then likely to show that distance = distance or $0=0$, a tribute to his impeccable arithmetic but of zero value to the problem at hand.

Instances of the laws

The next step is the interaction between the statement of the problem and the law(s) that have been selected. This generally means that special instances of the law(s) are exhibited to reflect the particular conditions of the problem. These are adjoined to the set of equations that are mathematical expressions of the relationships enunciated in the statement of the problem. If we let subscripts distinguish the two trains, this latter set of equations is:

$$D_2=D_1$$
$$r_2=r_1+16$$
$$t_1=6$$
$$t_2=4.$$

After making the substitutions in the law, the following set of equations can be added:

$$D_1=r_16$$
$$D_2=(r_1+16)4.$$

The technical operations

It is now time to manipulate. As every teacher knows, blind manipulation is not mathematics and will seldom lead to the solution. Even more important in mathematics than knowing how to do something is knowing what you want to do in the first place. If the present problem has been understood, the student knows that he is seeking values for r_1 and r_2. His manipulation must proceed towards this end, and his proficiency in achieving the goal is a measure of his manipulation ability.

Exhibiting the answer

If the solver knows where to stop the manipulation, it is only because he has the

interpretation of the symbols constantly before him. There will not be any problem concerning the inverse mapping, i.e., once he has decided that he is going to cease manipulation, when he has expressions for r_1 and r_2:

$$r_1 = 32$$
$$r_2 = 48,$$

he will have no difficulty in stating: "The velocity of the first train is 32 miles per hour, and the velocity of the second train is 48 miles per hour."

The above treatment undoubtedly has limitations. For one thing, not every problem is as amenable to analysis as the illustrative example. Variations and failures will occur. But the author has successfully employed the general approach to reduce the tensions usually associated with "word problems." It is not proposed that the teacher should discuss psychology or mappings with an elementary algebra class. However, much of the apprehension that "word problems" induce can be dissipated if the essential nature of word problems is discussed: (a) when they are "word problems" and when they aren't; (b) when they become problems that require proficiency in the discipline of mathematics and cease, thereby, to be problems of physics, economics, sociology, etc.; and (c) what kinds of decisions the potential solver must make that are not required in a nonword problem. With the current emphasis on logical analysis and the study of structure, the role of the "word problem" should be re-evaluated. Carefully presented, it can be a ready complement to this emphasis, an ample justification for its retention.

Letter to the editor

Dear Editor:

The following alternate solution to a problem discussed by Professor Lewis in the November, 1960, issue of THE MATHEMATICS TEACHER might be of interest.

Problem: Find all of the ways a dollar can be changed into 21 coins.

Let P, N, D, Q, H denote the number of pennies, nickels, dimes, and half dollars, respectively. Then P is a positive integral multiple of 5 and it is easy to see that P must be less than 20. We treat four cases as follows:

		$P=$ 0	5	10	15
(1)	$N+D+Q+H=$	21	16	11	6
(2)	$5N+10D+25Q+50H=$	100	95	90	85
(3)	$N+2D+5Q+10H=$	20	19	18	17
(4)	$D+4Q+9H=$	-1	3	7	11
			from (1) and (3).		

$P=0$ is impossible since $D+4Q+9H=-1$ is impossible.

$P=5$ implies [using (4)] that $H=Q=0$, $D=3$, and [using (1)] $N=13$.

$P=10$ implies that $H=0$ and $Q=0$ or 1. If $Q=0$, then $D=7$, $N=4$. If $Q=1$, then $D=3$, $N=7$.

$P=15$ implies that $H=0$ or 1. If $H=0$, then $Q=2$. [$H=Q=0$ would imply $D=11$; $H=0$, $Q=1$ would imply $D=7$. These are impossible in view of (1).] If $H=0$, $Q=2$, then $D=3$, $N=1$. If $H=1$, then $D+4Q=2$ and $Q=0$, $D=2$, $N=3$.

The five ways to change a dollar into 21 coins are tabulated as follows:

P	N	D	Q	H
5	13	3	0	0
10	4	7	0	0
10	7	3	1	0
15	1	3	2	0
15	3	2	0	1

This brings to mind a problem which I first met and solved as a ninth grader. A farmer bought 100 animals consisting of cows, pigs, and sheep—one or more of each—for a total of 100 dollars. If cows are 10 dollars each, pigs are 3 dollars each and sheep are 50 cents each, how many of each did the farmer buy?

Let C, P, and S denote the number of cows, pigs, and sheep respectively purchased. Then

$$C+P+S=100$$
$$10C+3P+\tfrac{1}{2}S=100$$
$$20C+6P+S=200$$
$$19C+5P=100$$
$$19C=5(20-P).$$

Then $19C$ is a multiple of 5, $19C<100$, and it follows that $C=5$, $P=1$, and $S=94$. And so the farmer bought 5 cows, 1 pig, and 94 sheep.

Sincerely yours,
LAWRENCE A. RINGENBERG
Eastern Illinois University
Charleston, Illinois

Analysis of a spider nomograph

JAY M. JEFFERY, *Charles F. Brush High School, Cleveland, Ohio.*

An additional analysis of the spider nomograph published in a previous issue of this journal.

THE RECENTLY DESCRIBED "spider" nomograph* is a convenient device for solving reciprocal sum equations. By means of analytic geometry, it can be demonstrated that an equation like $1/a+1/b=1/c$ may be expressed by such a diagram. In addition, this approach provides a means for determining the angles and unit lengths of other nomographs of the same type, which also solve reciprocal sum equations, but do so with an expanded answer scale.

The first problem then is this: Is it possible to represent the equation $1/a+1/b=1/c$ by a "spider" nomograph (Fig. 1) whose legs form 60° angles and whose units are all equal but are of arbitrary length? To answer this question, the initial step is to construct a mathematical model in the xy-plane. In Figure 2 the nomograph consists of three legs: OA, OD, and OC. The "spider" has been placed so that the intersection of the legs is at origin, and line OA lies along the X-axis. The line AC represents a straightedge placed on points A and C, cutting OD at point B. Angles α and β are restricted in the following ways: $0°<\alpha<180°$ and $0°<\beta<\alpha$. The length of OA equals a. The length of OC equals b. Both a and b are arbitrary positive values. Points C and D could, of course, be in quadrants I or II, depending on the choice of angles.

Second, the co-ordinates of the points A, B, C, and O are to be determined. They may be found for points A, C, and O by inspection. Point A is $(a, 0)$. Point C is $(b \cos \alpha, b \sin \alpha)$, and point O is $(0, 0)$.

* Lowell T. Van Tassel, "Notes on a 'Spider' Nomograph," THE MATHEMATICS TEACHER, LII, (November 1959), 557–559.

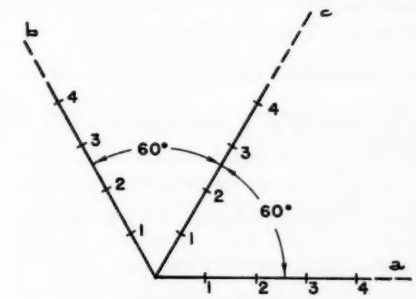


Figure 1

Finding the co-ordinates of point B is more difficult and is tantamount to solving the problem. To do this, the equations for the line through points O and D and the line through the points A and C are determined. Then, these equations are treated as a pair of simultaneous equations, and they are solved for the intersection point B. As a result, the distance from point O to point B is equal to the x-co-ordinate of the point B divided by the cos β. This distance on the line OD represents the

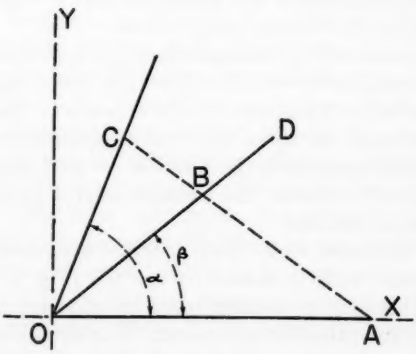


Figure 2

solution on the nomograph and should be the length c.

By inspection, the equation for the straight line through points O and D is

$$y = x \tan \beta. \tag{1}$$

The equation for the straight line through points A and C is found by means of a basic formula of analytic geometry which requires that two points of the line be known. Thus, the line is

$$(b \sin \alpha)x + (a - b \cos \alpha)y = ab \sin \alpha. \tag{2}$$

By substitution, equations (1) and (2) become

$$x = \frac{ab \sin \alpha}{a \tan \beta + b(\sin \alpha - \cos \alpha \tan \beta)}, \tag{3}$$

which may also be written

$$x = \frac{ab(\sin \alpha / \tan \beta)}{a + b(\sin \alpha / \tan \beta - \cos \alpha)}. \tag{4}$$

The distance OB is now obtained by dividing the x-co-ordinate by the cos β. Thus,

$$OB = \frac{ab(\sin \alpha / \sin \beta)}{a + b(\sin \alpha / \tan \beta - \cos \alpha)}. \tag{5}$$

For arbitrary choices of α and β within the given limits, if $OB = c$, then

$$c = \frac{abE}{a + bF}, \tag{6}$$

where E and F are constants, and $E = \sin \alpha / \sin \beta$, $F = \sin \alpha / \tan \beta - \cos \alpha$.

If $E = F = 1$, then statement (6) becomes $c = ab/(a+b)$ or $1/a + 1/b = 1/c$. Also, $\sin \alpha / \sin \beta = 1$ and $\sin \alpha / \tan \beta - \cos \alpha = 1$. The attempt to solve the two trigonometric equations yields the solution $\alpha = 120°$ and $\beta = 60°$. Hence, the desired results have been obtained.

Suppose now that a more expanded linear scale is desired on line OD (Fig. 2). This may be accomplished by choosing a larger value for the constant E in equation (6) and a value of 1 for F. Equation (6) then becomes

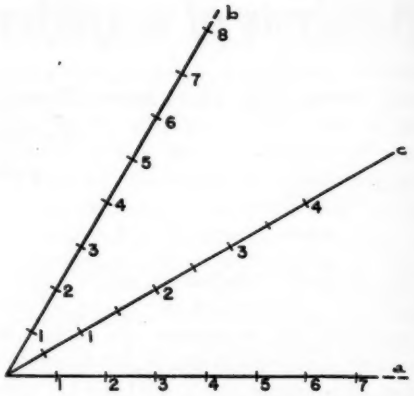


Figure 3

$$c = \frac{abE}{a + b}. \tag{7}$$

It is clear from equation (7) that the length of each c is increased by the factor E.

The value of E must be greater than 1 to achieve any useful expansion of the scale. However, E must be less than 2. If $E = 2$, then $\alpha = \beta = 0°$, and the "spider" is gone. If E exceeds the value 2, then real values for α and β cannot be determined.

A convenient choice for E is the square root of 3. If $E = \sqrt{3}$, then

$$\frac{\sin \alpha}{\sin \beta} = \sqrt{3}. \tag{8}$$

Also, since $F = 1$, then

$$\frac{\sin \alpha}{\tan \beta} - \cos \alpha = 1. \tag{9}$$

In solving equations (8) and (9), the result is that $\alpha = 60°$ and $\beta = 30°$. Thus, a "spider" nomograph may be drawn on which the unit lengths of the central leg have been expanded (Fig. 3). The expansion permits a greater degree of accuracy in reading answers. Also, the choice of $E = \sqrt{3}$ allows construction of the nomograph by compass and straightedge. The unit lengths on legs a and b are arbitrary and equal. The units on c are equal to the square root of three of the units used on a and b.

• EXPERIMENTAL PROGRAMS

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

Summer seasoning at Kenyon College

by Daniel T. Finkbeiner, Kenyon College, Gambier, Ohio

Introduction

A learned nonagenarian of my acquaintance never lost his interest in gardening; his acreage shrank steadily with advancing years, but he took pleasure in cultivating at least an herb garden. "In youth," he would quote, "a man sows wild oats. Later on, he grows sage."

Our society, it seems, is accustomed to permitting its youth a reasonable portion of wild oats, but only recently has serious attention been given to the choice of spices and herbs to flavor the educational menu of even our most promising young scholars. Indeed, much of the recent debate within American education has focussed upon the basic ingredients which constitute a hearty, substantial, and balanced diet for the full development of the mind and spirit of our future leaders. It is not my purpose here to enter that controversy nor to prescribe a recipe for secondary education. Rather, I seek merely to describe an educational experiment, the object of which was to capture the flavor of mathematics and to whet the appetites of students who had already demonstrated unusual digestive capacities.

Although public sentiment in support of curricular revision is commonly ascribed to post-Sputnik alarm, several large-scale programs of academic enrichment have been developing during the past decade. Consequently, the educational community was prepared to respond immediately and constructively to the newly aroused public desire for excellence in American education. When more funds, both public and private, were pledged to the goal of excellence, there was no lack of sound experimental programs which had been developed successfully in pilot studies and were ready for expansion to national proportions. Particularly in mathematics special programs had been developed by the Ford Foundation's School and College Study for Admission with Advanced Standing, the College Board's Commission on Mathematics, the Mathematics Association's Committee on the Undergraduate Program, the National Science Foundation's School Mathematics Study Group, and many others.

One of the most promising new experiments is the National Science Foundation's Summer Science Training Program for High Ability Secondary School Students. Broadly conceived, this program is designed to provide the superior high school student with tuition-free summer education in science beyond the scope of the normal high school curriculum. During 1960 grants were awarded under this program to 137 institutions, offering a variety of unusual learning opportunities to over 7,000 boys and girls. Lengths of the courses ranged from three to ten weeks; some proposed study in a single science, others in several; some were residential programs, others for commuting students only.

The institute which I describe here was

an intensive, four-week residential program in mathematics conducted at Kenyon College. Our principal aim was to stimulate imagination and creativity, particularly in mathematics. Since its rural location in central Ohio makes Kenyon an unusually attractive place for uninterrupted summer study, we felt that we could best accomplish our aims in a short, concentrated course, allowing the students enough time after the program for vacationing and further independent exploration of mathematics.

SELECTION PROCEDURES

Well aware that such a program could be effective only if we brought an exceptional student body into close contact with experienced and able teachers, we sent announcements and application materials directly to the chairmen of mathematics departments in 900 secondary schools in the northeast quadrant of the nation, asking them to nominate any student who had truly exceptional promise as a mathematician. Application forms were mailed also to nearly 500 individuals who had learned of our program through the N.S.F. announcement or through the public news media.

Each applicant was required to submit a completed application form and to obtain two letters of recommendation, one from a mathematics teacher and the other from a teacher of science, English, or foreign language. Transcripts, test scores, and other supporting evidence were suggested but not required. A scholarship form, indicating the degree of financial need and signed by a parent, was required from any applicant who requested funds for the support of part or all of his costs for room, board, and travel. To assure that benefits of the program would be transmitted to the high school classes by the participants, graduating seniors were declared ineligible.

Completed applications and all supporting data were received from 352 students in 31 states; three-fourths of the candidates were boys. Selection was based upon careful evaluation of the applications by a committee of three mathematicians, with major weight being given to the teachers' letters of recommendation and evidence of unusual mathematical activity and intellectual vigor. Test scores played a secondary role. As expected, nearly every candidate ranked in the upper tenth of his class; transcripts generally were covered with A's and did not serve to discriminate between degrees of excellence. The committee screened all applications to obtain a group of 130 finalists. A second screening produced lists of 50 participants and 50 alternates. Eventually, 38 boys and 13 girls from 20 states attended the program; there were 39 juniors, 11 sophomores, and 1 freshman.

ACADEMIC PROGRAM

In order to challenge these extremely able and well-prepared students, we planned a very demanding academic program, prepared to reduce the pace if necessary. It was a pleasant surprise to learn that we had not overestimated these students, and we actually supplemented our original syllabus, covering in four weeks material which is fully equivalent in content and rigor to the first-year course for nonscience students at Kenyon.

The teaching staff consisted of two Kenyon mathematicians (Wendell D. Lindstrom and myself) and two secondary-school mathematics teachers (Donald H. Byerly, now of the Westtown School, and Miss Maita Levine of Woodward High School in Cincinnati). Both had previous experience with the Advanced Placement Program and N.S.F. Summer Institutes for teachers. In the Kenyon program they served as resident counselors in the two dormitories, conducted scheduled study sessions, and instructed informally at all hours of the day and night. The college teachers were responsible for class lectures, independent study, and conferences during periods reserved for supervised study.

A typical daily schedule included two hours of course lectures in the morning and a two-hour, supervised study session in the afternoon, followed by a two-hour period for recreation. Three evening hours were reserved for independent study and special events. Classes met six days a week, but the supervised study session was omitted on Saturday. For the morning lectures, the group was divided into two sections which were taught in parallel, with one college and one high school teacher assigned to each class. In order that all staff members could observe all students, the high school teachers alternated sections weekly, and the college teachers exchanged sections after two weeks.

In selecting the mathematical content of the course, we established several criteria: it should not duplicate work which would normally be elected by these students in high school or college; it should be good mathematics, providing insight into basic ideas and fundamental methods of thought; it should be fresh and stimulating, suggesting interesting topics and problems for further independent investigation. The program which we actually developed can be described under six categories:

1. The *central course* included formal logic, set theory, probability theory, and matrix algebra. This material comprises the first five chapters of *Introduction to Finite Mathematics* by Kemeny, Snell, and Thompson, which we used as a text, supplemented according to the interests and abilities of the class. Daily problem-assignments were given and solutions were written in problem notebooks which were read once each week by the staff.

2. Special *staff lectures* were presented as follows: three hours on group theory; three hours on set-theoretic formulations of relations, functions, operations, and abstract systems; one hour on finite geometry.

3. *Guest lectures* were presented by visiting mathematicians and included two hours on Boolean algebra, one hour on foundations of analysis, one hour on information theory, six hours on machine computation and programming, including observation of an IBM 650 in operation, solving a problem which had been programmed by the class.

4. *Student talks* were presented on topics of independent study: combinatorial topology, Diophantine equations, basic language programming for the IBM 650, inversion and linkages, and non-Euclidean geometry.

5. *Written reports* on at least two topics of current mathematical research were required of each participant as a special assignment intended to acquaint him with the nature and diversity of modern mathematics. Each student was asked to begin by reading recent issues of *Mathematical Reviews*, selecting topics of interest to him. He then was to use the mathematics library to find introductory treatments of these topics and to write a concise synopsis of what he had learned. Although this assignment met with initial dismay and confusion, enthusiasm soon developed. Many students commented that this proved to be one of the most valuable experiences of the program, and the quality of their reports supports this opinion.

6. *Kenyon Faculty lectures* on nonmathematical topics were scheduled to present a diversity of intellectually challenging ideas. Each lecture was followed by a period for discussion and argument. Titles included "Is America a Political Failure?", "The Van Allen Belts," "Science and Poetry," and "La Chimie dans l'Espace."

Evaluation procedures

Hoping to create an atmosphere of natural curiosity and independent thought, we told the students at the beginning that

the program was intended as an experience in learning, free from the specter of examinations and grades. This was a calculated risk, but we assumed that each member of this exceptional group was sufficiently motivated by intrinsic interest in mathematics. It is gratifying that our assumption was justified; with only a few exceptions, a high level of interest and perseverance was maintained throughout the course. In lieu of grades the staff in conference wrote a summary report on each student, based upon our observations of his initiative, originality, ability, tenacity, and progress.

In retrospect

Almost any academic program is guaranteed success when the class is composed of unusually talented, enthusiastic, and serious young people. In contrast, any program which seeks to arouse creative thought and independent study can never claim unqualified success. In these concluding remarks I shall try to evaluate various aspects of our program and to suggest changes which might have brought our efforts somewhat closer to our intended purpose.

Our admissions procedure seems to have been successful. The students who attended were bright, eager, and likable; they quickly established a strong esprit de corps and enjoyed living together and exchanging disparate views. Undoubtedly there were other applicants, equally capable, who were not chosen in our top fifty because of the extreme difficulty of distinguishing between excellent students from all over the nation, often with contrasting backgrounds of educational opportunity. We chose to rely heavily upon the opinions of teachers who responded candidly to our request for a frank appraisal of the quality of mind of the candidate, describing weaknesses as well as strengths.

Likewise, we feel that the topics selected for study had the qualities which we wanted—basic mathematical content with a modern flavor and freshness, capable of being treated intuitively at first and with increasing abstraction as the course progressed. The special lectures by members of the staff, visiting mathematicians, and Kenyon professors in other subjects successfully relieved the central course and introduced a welcome variety of challenging ideas in mathematics, literature, and political philosophy, as well as in science.

The decision to stress learning rather than grades was correct, we feel. In a program of this nature grades and academic credit should play no part. However, the staff agreed that it might have been wise to give one written assignment each week to be worked individually without help and then read critically by the instructors, serving as an indication of individual progress.

My greatest source of dissatisfaction arises from the conviction that a few of these young mathematicians were capable of far more independent work and original "research" than we were able to develop within the four weeks. Perhaps this is inevitable in a short program. I can report with strong assurance that I have never seen a staff of teachers work so hard, so intensely with their students, nor have I seen classes with more enthusiasm and curiosity. Yet, the program would have been more effective had we been able to direct individual study projects from the first day. The student talks, presented in the final week, should have been scheduled earlier in order to encourage other students to investigate similar topics. Of course, the direction of individual study is very time-consuming for the staff. Since the teachers worked perilously close to the point of exhaustion to conduct the program described above, a more strenuous effort at individual direction would surely have required one additional teacher with a broad knowledge of mathematics.

In summary, our experience with Kenyon's first Summer Program in Mathe-

matics was most encouraging; we feel that we realized the objectives of the program to a far higher degree than we had thought possible.

The students of this exceptional quality are able and anxious to undertake serious study in mathematics and to originate promising efforts in its creative aspects. To both staff and students this program has been an exciting experiment in education; as this and similar experiments are continued and improved, their contributions will surely be felt throughout the schools and colleges of the country and, ultimately, in the strength of the human resources of the nation.

The high-ability students' institute at Flagstaff

by J. H. Butchart, Arizona State College, Flagstaff, Arizona

Beginning in 1957, Arizona State College at Flagstaff has sponsored summer institutes for high school teachers, financed by the National Science Foundation, but, until the fall of 1959, we were unaware that the N.S.F. was also interested in offering extra study opportunities to high school students. An educator in the Phoenix school system, Dr. Fred Bedford, called our attention to the possibility and urged us to make out a proposal for a math camp, modeled on the one at Florida State. He reasoned that Flagstaff, with its summer resort climate in the pines of northern Arizona, would have more appeal for the gifted students in Tucson and Phoenix than either of their home cities.

Our proposal was, of course, rather carefully guided by the directives of the foundation. The math camp was to be of five weeks' duration, and the thirty boys and girls participating would have just finished their junior year in high school, would have taken math each year, and would have an IQ of 120 or over. Of the three types of institutes, we preferred that in which the students mainly study material which they are not likely to encounter in the traditional course offerings. They would not receive high school or college credit, and attendance at the institute would not be the basis for omitting any traditional course. (The other types of institutes are those in which the students study standard courses for credit and will, therefore, be given advance placement and those in which the participants act as research assistants to recognized scientists, who are seldom, if ever, mathematicians.)

Dr. V. M. Gillenwater, executive dean of the college, aided by Dr. Bedford, did most of the preparatory work in connection with the proposal. I was listed as director, but my work was to be only part time, as I was to teach in our institute for high school teachers. After our proposal was accepted, I attended the conference for directors at Albuquerque.

Our principal instructor in the institute, Mr. Henry Nicholson, a professor at Phoenix College, had been a participant in the institute at the University of Kansas and had studied the operation of the Illinois system for teaching secondary students. He had also become an expert in

the material presented by Kemeny, Snell, and Thompson in *Introduction to Finite Mathematics*. It was difficult to find a second instructor, but we finally secured Mr. Robert Broderick, who is responsible for supervising mathematics instruction in the public school system of Glendale, Arizona.

The selection of participants was aided by Dr. Gillenwater's foresight in specifying that only two students should apply from the same high school and that applications should be forwarded from high school teachers or administrators rather than from students themselves. This had the effect of putting more than half the responsibility for the selection of the students on officials familiar with them. About 250 students asked for application forms, and 95 mailed them in. An important criterion—besides grades, standardized tests, and testimonial letters,—was the student's own composition on why he was interested in attending such an institute. Many of these youngsters told about their projects for science fairs and stated that they hoped to be challenged by meeting and competing with other superior students.

Some of the selected applicants had already accepted bids from other institutes by the time they received ours, but our final choice included 8 girls and 21 boys, 19 of whom were from Arizona and 10 from elsewhere. The boys were housed in a college dormitory under the supervision of Mr. Herbert Swanson, a staff member of the local college, while the girls lived in another dormitory under the supervision of Miss Dixie Dozier, a school administrator from Yuma, Arizona.

The two instructors started the ball rolling on the first day by giving the Phoenix College placement test for incoming freshmen. This is somewhat more difficult than the usual freshman test and is designed to indicate whether a student is capable of entering directly a course in analytic geometry and calculus. The group had a median score significantly higher than that of the high school teachers in the adult institute to whom I gave the same test. The best score made by a student was 42 out of a possible 45, while the best by a teacher was 44; but the median for the students was 29 and for the teachers 22. This may be explained by the fact that some of the teachers were from small school systems where they had to teach mathematics without proper training in the field.

It was inspiring to work with these exceptional students. Mr. Nicholson and Mr. Broderick conducted two formal class periods in the morning, an afternoon session with varied subject matter, and an evening seminar each day. Between periods they were accessible for consultation. Regular assignments were given and graded. In addition, the instructors brought to the attention of the students special problems from such sources as the *Mathematical Monthly* or the *Mathematics Magazine*. One that stirred up a good deal of interest concerned the number of zeros at the right end of the expanded form of 10,000! The students had evidently read far beyond the call of duty, as I found when I proposed a problem in probability. One student had heard all about it from the *Scientific American*. It was a revelation to see what such a group can do when there is no ceiling set by the average students, whose pace determines the speed of the ordinary class. They learned about Venn diagrams and truth tables easily.

Each participant had his own project, a chart concerning some problem, such as in circuit analysis or probability. The college conducts a regular Thursday evening forum which is usually addressed by an outside speaker. The high school institute was given the responsibility of presenting the program for the forum at its last session for the summer.

The students found time to prepare some very interesting displays, such as a comparison between the truth table method of analyzing a complicated chain of logical statements from Lewis Carroll and the same problem simplified by Euler

circles. One student presented the problem concerning the probability that in a group of people two individuals share the same birthday. A network of switches with lights that went on and off was especially well received.

Flagstaff is located in a very scenic area, and the adults responsible for the institute thought it would be a shame to let the students, especially those from out of the state, leave without the opportunity to see the natural attractions. A trip to the Grand Canyon was arranged, for which the students paid for their transportation on the college bus. The counselors and director and other interested persons furnished transportation in private cars to Sunset Crater, Lowell Observatory, Walnut Canyon, and Oak Creek Canyon. During the last week the whole group enjoyed a picnic under ponderosa pines between a trout stream and the red cliffs of Oak Creek Canyon.

All in all, we felt that the National Science Foundation has the right idea in paying half the students' way so that they can receive the benefits from such an institute.

Intermediate algebra for high-ability high school students

by Berneice E. Brush and Edith Hunt, Billings, Montana

For two summers, 1959 and 1960, thirty-one mathematically talented students have been able to advance in their regular mathematics curricula and, at the same time, gain mathematical knowledge of topics not usually presented to the high school student. Eastern Montana College of Education, with the co-operation and assistance of the National Science Foundation, has provided an eight weeks' summer program in intermediate algebra for the high-ability high school student who has completed his sophomore year.

The objectives of this program have been:

1. To provide a challenge and a means of growth for the student who possesses both mathematical talent and interest;
2. To provide an accelerated program in intermediate algebra with stress on algebraic structure and concepts, so that the student may develop a greater understanding of mathematics;
3. To serve as a demonstration class for interested secondary-school mathematics teachers.

Selection

Participation in this, a commuting program, was necessarily limited to students from the schools in this area.

Therefore, selection of the student participants was made under the supervision of Sister Mary Lenore, Principal of Central Catholic High School; Willard Wallace, Principal of Billings West High School; Charles E. Borberg, Principal of Billings Senior High School; and Mr. Bernard Hughes, Superintendent of Huntley Project Schools. This group, at a meeting with Director Oliver Peterson, agreed that

selection should be based upon general aptitude, reading ability, interest in the course, mathematical aptitude, mathematical achievement, and the recommendation of his mathematics teacher.

Program

The daily program consisted of two hours' instruction in intermediate algebra, with stress on structure and algebraic concepts, and an additional hour of lecture and discussion of various topics in science and mathematics.

While the text used by the students was *Algebra for Problem Solving* by Freilich, Berman and Johnson, the actual class instruction followed the materials in *Intermediate Mathematics* as developed by the School Mathematics Study Group.

Below is an approximate time schedule on the topics covered in *Intermediate Mathematics:*

Chapter 1. The Real Number System (Two weeks)
- a. Construction of Number Line and Real Numbers
- b. Development of the Number System
- c. Factoring
- d. Order Relations and Equivalence Relations

Chapter 2–3. Linear and Quadratic Functions (1½ weeks)
- a. The Function Concept
- b. The General Linear Equation and Systems of Equations
- c. Variation
- d. Co-ordinate Geometry (Brief)
- e. The Quadratic Function

Chapter 5. Complex Number System (One hour for one week)
- a. Development by Ordered Pairs

Chapter 8. Logarithms and Exponents (One hour for one week)
- a. Logarithmic and Exponential Functions

Chapter 4. Quadratic Equations (One week—two hours)
- a. Methods of Solving
- b. Role of Discriminant
- c. Reducing Equations to the Quadratic Form
- d. The Remainder Theorem

Chapter 6. Equations of the Second Degree (One week—two hours)
- a. Parabolas, Circles, the Ellipse, and Hyperbola
- b. Solution Sets
- c. Systems of Inequalities.

Chapter 11. Sequences and Series (One week—two hours)
- a. Arithmetic and Geometric
- b. Binomial Expansion

Chapter 10. Permutations, Combinations, and Binomial Theorem (One week—one hour)
- a. Binomial Formula

In addition, the definition of a field, the idea of a one-to-one correspondence, the use of counter examples, and deductive proofs in algebra were introduced.

The staffs of the science and mathematics departments at Eastern Montana College of Education gave demonstrations and lectures during the third hour of the daily program. Topics discussed by the professors of science included: Genetics, Elementary Electrochemistry, Electronic Physics with a demonstration of Millikan's Experiment to Measure Planck's Constant, and Ecology. The lecture on Ecology was followed by a field trip to a local pond to observe and collect specimens.

Lectures in mathematics covered Symbolic Logic and Boolean Algebra, Elementary Group Theory, Modular Arithmetic, Finite Geometry, Matrices, High-Speed Electronic Computers, and Statistics and Probability. Many of these topics required from three to seven hours for presentation.

Too, each student was required to do some reading and research on some mathe-

matical topic. Miss Edith Hunt, assistant instructor, aided the students in finding materials and in organizing their papers. Several students constructed displays and models as a part of their project. Representative projects were: Napier's Bones, A Simple Computer, Topological Surfaces, Geometry of Four Dimensions, Non-Euclidean Geometry, Infinity—a Tool of Mathematics, and the History of Mathematics.

The group had two field trips during the eight weeks. The first was in connection with the lectures in ecology, and the second took them to Montana State College to observe the IBM 650 in operation. Professor Peterson's lectures on Boolean algebra followed by Dr. Young's lectures on computers gave the participants sufficient information to understand the functioning of the IBM electronic computer.

Observation and follow-up

Two features of the program deserve special mention. First is that of observation. Those students enrolled in mathematics courses and any teachers in the area could, if they wished, visit the class and observe high school students being taught the materials recommended by the Commission on Mathematics.

The second feature is that of follow-up. The board of trustees of each of the participating schools has approved the course and granted one-half credit toward high school graduation. Students from Billings Senior High School and Billings West High School continue as an accelerated group. This year, these students will complete work in college algebra and trigonometry and will take Mathematics 13 as seniors. The text presently used in Mathematics 13 is *Introduction to College Mathematics* by Milne and Davis.

Evaluation

The reports on the follow-up would indicate that the programs have been successful. It is hoped that the number of participants might be increased if such a program were available. Also, students from a greater commuting radius would be accepted. This might mean that there would be changes in the method of selection.

The range in IQ's for the 1960 group was from 112 to 140 with a mean IQ of 123.

The group was given the Co-operative Intermediate Algebra Test at the beginning of the course and again the last week. The median range in percentile ranks increased from the 62nd percentile on June 13th to the 89th percentile on August 3rd. The median on the U.S.A.F. College Algebra—American Council of Education Test, given on August 5th, was 69.

All students felt that the course met or surpassed their expectations. Following are typical comments of students and parents:

"The features of the program I like most are the speed in which we cover things and the extra material we get through lectures. I like the fact that we do projects the least."

"This is the first time some of us have really had to work. I think the best part of the whole course is having the special lectures and meeting people who have made their life work in some special branch of mathematics."

"This course has been more of a challenge to me. I feel that I have learned more, for there was more theory instead of being so mechanical."

"As parents of a high school age student we appreciate the type of work this class has offered. In addition to broadening the base for college entrance, it also provided training in independent thinking and study habits needed by many students who plan to enter college. The sampling of various scientific subjects will help students pick their field with more confidence."

Credit for the success of the program should go to the two directors, Mr. Ed. Dolney and Mr. Oliver Peterson; to the visiting professors, Dr. Adrien Hess and Dr. Young; and to the staff at Eastern Montana College of Education.

A research program for gifted secondary-school students

by Albert Wilansky, Lehigh University, Bethlehem, Pennsylvania

1. With the support of the National Science Foundation, a Junior Research Seminar was held at Lehigh University for six weeks during the summer of 1960. The seminar met from nine to twelve five mornings each week. There were 19 boys and 6 girls, chosen from almost 200 applications. One student had just completed 9th grade; three, 10th grade; twelve, 11th grade; nine, 12th grade. All had at least two semesters of algebra and one of geometry. Selection was on the basis of academic record, particularly in mathematics; teachers' comments about the students' initiative and ability; activities such as reading and participation in clubs and science fairs; academic honors, both local and national; performance in the Mathematical Association of America contest; performance on standardized tests.

All the participants were clearly gifted. On 20 transcripts under IQ were listed numbers ranging from 123 to 143, with 130 as median. It turned out that the participants were also gifted in music, athletics, and leadership.

There was a lecture each morning from nine to nine-thirty. For the remaining two and one-half hours, the students investigated projects which had been outlined in the lectures. Some of the more outstanding students were given reading material which carried them further and suggested more topics for investigation, or which introduced them to topics not mentioned in the lectures. But, in all cases, the emphasis was on questions. Students who asked good questions were praised and even rewarded by mention in the "Proceedings," an informal journal which appeared piecemeal on the notice board. The same reward went to those who achieved interesting results. As indicated below, the questions asked in lectures were usually ill-defined. The students had both freedom to choose the topic on which they would work and responsibility to formulate the questions which they would answer. In this, I believe, lies the essence of research. Very often, two students working on the same topic would compare notes and find that while one had been solving equations, the other had been drawing the necessary diagrams.

From nine-thirty to twelve I interviewed the students in a separate room, mostly one by one, occasionally in groups of two or three. In a 15-minute chat they told me their activities, and I satisfied myself that they had definite plans for the immediate future, if necessary making suggestions to ensure this. I took notes during these interviews. They came to me in alphabetical order; hence, I saw each student two or three times a week during the course of the program.

Meanwhile, back in the classroom my assistant, Mr. I. D. Berg, was available for instant discussion. He was approached by students as they felt the need. He answered questions about the lectures, criticized the students' work, and made suggestions. Much of the success of the seminar was due to Mr. Berg's judgment, discernment, and skill in these contacts. He and I listened to each other's lectures but exerted no other influence on each other.

The experience was, for us, unique, intensely stimulating and rejuvenating, and infinitely enjoyable. In a closing evaluation, each student expressed enthusiasm.

Selected puzzles, mostly from the problem section of the *American Mathematical Monthly*, were posted, but not discussed in class time.

2. The topics of the seminar were (1) Linearity, (2) Multiplication, (3) Residue Classes, (4) Group Theory, (5) Miscellaneous.

Abstraction for its own sake was not pursued; when abstractions arose, they were studied in the concrete framework in which they arose. For example, the class of 2 by 2 matrices is an algebra. Question: Can a matrix have two inverses? Most students answered no because they had obtained a specific formula for the inverse. Only a very few produced the argument, if $AB=BA=AC=CA=I$, then $B=(CA)B=C(AB)=C$. It was pointed out to them that this latter argument held in any algebra (A list of postulates for an algebra had been posted.), several of which had already been mentioned. The same is true for the proposition, if $AB=O$ and $B\neq O$, then A^{-1} does not exist. On the contrary, any proof that $AB=I$ implies $BA=I$ must use nonalgebraic properties of matrices, since there exists an algebra in which it is false. The students found some such proofs.

Rigor was not emphasized. If a student had good reasons for believing the truth of his result, he was not pushed to give a proof. Proofs were given in the lectures, however.

3. The lectures on linearity went as follows: A *point* is a pair of numbers; if $P=(a, b)$, $Q=(c, d)$, then $P+Q$ is defined to be $(a+c, b+d)$; if A, B are sets of points, $A+B$ is defined to be the set of all $P+Q$, with P a member of A, Q a member of B.

Projects: Investigate circumstances under which $A+A\subset A$, $A+A\supset A$, $A+A\subset 2A$, $A+A\supset 2A$, $A+A+A=3A$, $aA+bA=(a+b)A$, $a(A+B)=aA+bA$, etc.

Many of the students had never studied analytic geometry, nevertheless, all drew axes. Some found before long that $aA+bA=(a+b)A$ for all positive, a, b if, and only if, A is convex. Others required guidance to arrive at this result. When the students said, if A contains O, then $A+A\supset A$, I asked if the converse was true. The set A consisting of the positive X-axis made its appearance. Some students doubted at first that $A+A\supset A$ because of what they called infinitely small numbers; however, the moment these ideas appeared they were rejected out of hand without a word from me. I seized the opportunity to discuss Bishop Berkeley's polemic on infinitesimals.

A student asked and later answered, if $A+A=2A$, must $A+A+A=3A$?

These ideas may be presented in one dimension, if desired, later, in two dimensions. All the questions about convex, linear, and affine subsets of a linear space retain their essential difficulty in the two-dimensional setting.

A basis is a pair of points Q, R, such that every point P is a linear combination of Q and R. *Project:* Find all bases. One girl did this in n-dimensional space, n unspecified (but finite).

The polar $\mathring{A}$ of a set A is the set of all (x, y), such that $|ax+by|\leq 1$ for all (a, b) in A. *Projects:* Find $\mathring{A}$ for $A=$line, circle, etc. (The polar of a circle may be any conic section.) Find $\mathring{\mathring{A}}$. Five students found that $\mathring{\mathring{A}}$ is the smallest convex symmetric set including A. One student found that $A=\mathring{\mathring{A}}$ if, and only if, A is the union of all sets B, such that $\mathring{B}=\mathring{A}$.

Linearity, part 2. Matrices: Two-by-two matrices were presented with their addition and multiplication. It was pointed out that the postulates for an algebra are satisfied. The students painfully verified the associative law of multiplication, and, hence, were in a good frame of mind to see the brief check, a few days

later, using summation notation.

Projects: Find identity for multiplication, and right and left inverses for a matrix. How many inverses may a matrix have? Does a matrix have a square root? (This last request generated a lot of excitement because the staff neglected to investigate it in advance.) I asked for "the four square roots of the identity matrix." Before long, students were reporting more than four square roots and soon one of them noted that for all a,

$$\begin{pmatrix} 1 & a \\ o & -1 \end{pmatrix}^2 = I;$$

hence, an infinite number of square roots had been found. A student undertook without consulting me—I would have dissuaded him—to solve the equation

$$\begin{pmatrix} u & v \\ w & x \end{pmatrix}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for u, v, w, and x. He succeeded after enormous labor, but his results were not instructive. I then decided to try it. Let $D = ux - vw$, so that $D^2 = ad - bc$. Let $t = u + x$, the so-called trace. Then

$$\begin{pmatrix} u & v \\ w & x \end{pmatrix}^2 = \begin{pmatrix} ut - D & vt \\ wt & xt - D \end{pmatrix}.$$

Hence, we must solve the equations $ut - D = a$, $vt = b$, $wt = c$, $xt - D = d$.

Adding the first and last of these yields $t^2 = a + d + 2D$, and this gives t, since $D^2 = ad - bc$. Then u, v, w, x are gotten from the earlier equations. In most cases a matrix will have 4 square roots, but sometimes there will be others; for example, if $a = d = 1$, $b = c = o$, then $D^2 = 1$, and if we can make $D = -1$, we get $t = o$, and the equations tell nothing about u, v, w, x except $ux - vw = 1$, $u + x = 0$. Since there are infinitely many solutions to these equations, we get an infinite number of square roots of I with $D = -1$. There are, however, only two with $D = 1$, since then $t \neq 0$, and u, v, w, x are all determined when t is. Several students discovered the matrix

$$\begin{pmatrix} o & 1 \\ o & o \end{pmatrix},$$

which has no square root at all. One last remark, to treat u, v, w, x as the unknowns in the square root problem leads to an intractable mess, whereas choosing t and D as unknowns made the work simple. I remarked on this and mentioned the quadratic equation $ax^2 + bx + c = 0$ in which the great contribution of the Arabs was to treat

$$x + \frac{b}{2a}$$

as the unknown instead of x. (We shall return to the topic of square roots below.)

By defining $A\,P$, with A a two-by-two matrix and P a pair of numbers, thus, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad P = (x, y),$$

then

$$AP = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$
$$= (ax + by, cx + dy),$$

each matrix becomes a transformation of the plane, and since $B(AP) = (BA)P$, following one transformation by another is the same as transforming by the product.

The matrix

$$\begin{pmatrix} o & 1 \\ o & o \end{pmatrix}$$

carries the point (x, y) to the point $(y, 0)$. This trip may be made in two steps, $(x, y) \to (0, y) \to (y, 0)$. The first step is perpendicular projection on the Y-axis given by

$$\begin{pmatrix} o & o \\ o & 1 \end{pmatrix}$$

the second is a reflection in the 45° line given by

$$\begin{pmatrix} o & 1 \\ 1 & o \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} o & 1 \\ o & o \end{pmatrix} = \begin{pmatrix} o & 1 \\ 1 & o \end{pmatrix}\begin{pmatrix} o & o \\ o & 1 \end{pmatrix}.$$

A transformation A is called a projection if there exists a line L, such that for all P, AP lies on L; and for all P on L, $AP=P$. One sees immediately that L must pass through O and that there must be a line M such that $AP=0$ for all P on M; and for all P, the line joining P to AP is parallel to M. We use the terminology $A=(L, M)$.

THEOREM: A *is a projection if, and only if,* $A^2=A$.

Proof: Let L be the set of all P such that $AP=P$, M the set of all P such that $AP=0$, etc.

Projects: Given arbitrary L, M, find the projection (L, M). Which projections have L perpendicular to M? If A, B are projections, under what circumstances would AB, $A+B$, $A-B$ be projections?

An illumination is thrown on the curriculum by the reaction of students to the problem, discuss

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Several boys and girls *who had studied no analytic geometry* found that all points on the 45° line were left fixed. They then observed that the point $P=(4, 2)$ was carried to the point $Q=(3, 3)$. By careful plotting, they were led to the conclusion that this is a perpendicular projection. I said, "Is PQO exactly a right angle?" Typical answer, after a long pause, was, "I can use Pythagoras' theorem." The students then wished to measure the distances with a ruler but admitted that this also would be approximate. After another pause, "I can see that $OP^2=4^2+2^2$, $OQ^2=3^2+3^2$," then finally, "To find the distance PQ, I can draw a horizontal and a vertical line each of length one, and so $PQ^2=1^2+1^2$." The distance formula of analytic geometry—produced on need without a hint of its existence! One must consider the value of teaching it to students of this caliber.

The Cayley-Hamilton theorem was presented as follows: As noted above in the digression on square roots, if

$$A=\begin{pmatrix} u & v \\ w & x \end{pmatrix},$$

then

$$A^2=\begin{pmatrix} ut-D & vt \\ wt & xt-D \end{pmatrix}=\begin{pmatrix} ut & vt \\ wt & xt \end{pmatrix} - \begin{pmatrix} D & o \\ o & D \end{pmatrix}=tA-DI.$$

Thus, we have $A^2-tA+DI=0$. This formula is the Cayley-Hamilton theorem for 2×2 matrices. If m, n are roots of the so-called characteristic equation $z^2-tz+D=0$ we have $(A-mI)(A-nI)=0$, $m+n=t$, $mn=D$. Setting

$$E=\frac{A-mI}{n-m}, \qquad F=\frac{A-nI}{m-n},$$

we thus have $EF=0$, $E+F=I$, $nE+mF=A$. Finally, E and F are projections; this is seen as follows:

$$E^2=\left(\frac{1}{n-m}\right)^2(A^2-2mA+m^2I)$$

$$=\left(\frac{1}{n-m}\right)^2(tA-DI-2mA+m^2I)=E,$$

since $t=m+n$, and $D=mn$. This is the spectral decomposition of A. Applications of it are found in the following formulas:

$$A^2=(nE+mF)^2=n^2E^2+nmEF+nmFE+m^2F^2=n^2E+m^2F.$$

Similarly, $A^k=n^kE+m^kF$ for positive integer k. It is also true for negative integer k; for example, $A(n^{-1}E+m^{-1}F)=E+F=I$; hence $A^{-1}=n^{-1}E+m^{-1}F$. The same technique yields $\sqrt{n}E+\sqrt{m}F$ as a square root of A.

Concerning square roots of I, it was pointed out that $A^2=I$ if, and only if, $1/2(A+I)$ is a projection. Since any square

root of I is a reflection, this has an obvious geometrical interpretation.

Higher powers of A may also be computed by means of the Cayley-Hamilton formula $A^2-tA+DI=0$, proved above. For example, to evaluate A^4, we merely divide z^4 by z^2-tz+D. If the remainder is $r(z)$, then $A^4=r(A)$. *Project:* Find all matrices with $A^3=0$. Are there any such for which $A^2\neq 0$?

4. We multiply points in such a way that the distributive and associative laws hold. This can be done in many ways: for example, $(x, y)\cdot(u, v)=(xu-yv, xv+yu)$, or $(x, y)\cdot(u, v)=(xu, yv)$, or $(x, y)\cdot(u, v)=(xu, xv)$. The last multiplication is not commutative. *Project:* Find all possible multiplications. If we set $Q_1=(1, 0)$, $Q_2=(0, 1)$, then Q_i^2 is a linear combination of Q_1, Q_2 indeed, so is Q_iQ_j for all i, j. Let us write

$$Q_iQ_j=\sum_{k=1}^{2} e_{ijk}Q_k, \qquad i,j=1,2.$$

The multiplication is determined by the eight e numbers. The associative law yields 16 equations, some of which are redundant. If the multiplication is assumed to be commutative, there are only three equations which can be solved explicitly; the solutions are not unique, of course. Strangely enough, the noncommutative case is simpler.

THEOREM: *If the multiplication is noncommutative, for any* P, P^2 *is a multiple of* P.

Proof: Choose Q so that P, Q are linearly independent. Then $P^2=aP+bQ$. From $P^2P=PP^2$ we obtain $PQ=QP$, unless $b=0$.

Thus, in the noncommutative case, some of the e's are zero. The final result is that the only noncommutative multiplications are $(x, y)\cdot(u, v)=(ax+by)\ (u, v)$. The commutative multiplications do not fall into such a neat pattern.

5. J_6 is the collection of numbers $(0, 1, 2, 3, 4, 5)$, with $a+b$ and $a\times b$ defined to be the remainder after the ordinary sum and product are divided by 6. For example, $3+5=2$, $3\times 5=3$, $2+4=0$, $3\times 4=0$. The postulates for a ring are satisfied. J_5 on the other hand is a field, that is, a ring in which each nonzero element has a reciprocal (and in which the multiplication is commutative). Thus, in J_5, $2\times 3=1=4\times 4$. Thus, $2^{-1}=3$, $4^{-1}=4$. The set of all elements in J_m which have a reciprocal form a group under multiplication. *Project:* Which of them are cyclic? The students soon conjectured that when P is prime, the group, in this case the set of nonzero elements, is cyclic. Later, the difficult proof was presented that this is indeed true of any finite field. A great deal of effort was expended by students in computing generators for the J_m multiplicative groups with m not prime. A Diophantine problem is to prove that a has an inverse in J_m if, and only if, (a, m) are relatively prime. (The theory of linear Diophantine equations had been presented earlier.) The mysterious behavior of the system of linear equations $x+2y=a$, $2x+y=b$ was shown, in that in J_p, with p a prime other than 3, it has a unique solution, while in J_3 it has none or many solutions. The students tracked this down to the determinant.

The students were amused by the fact that in J_3, $x^2-x+1=(x+1)^2$ but in J_5, x^2-x+1 is not factorable. They followed this topic further and compared it with the fact that x^2+1 is irreducible over the reals but is equal to $(x+i)\ (x-i)$.

6. Lagrange's theorem was given, the order of a subgroup is a factor of the order of the group. A student conjectured that every factor is indeed represented by a subgroup. Permutation groups and problems in permutations were shown to illustrate the group concept. Several students wrote out the multiplication table for the set of permutations on four objects. Fermat's and Wilson's theorems were deduced from Lagrange's theorem.

7. Other topics were attacked by various students. These were Egyptian fractions, Pythagorean triples, finite geometry, two-dimensional topology, inequalities.

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

The first mathematical dictionary

by Walter Crosby Eells, Washington, D.C.

The "first mathematical dictionary ever published in any language," according to the well-known secondhand bookdealers, Henry Sotheran and Co., of London, was Moxon's *Mathematicks Made Easie, Or, A Mathematical Dictionary*, published in London in 1679.

The writer has in his possession a well-preserved copy, bound in contemporary calf, of the second, or revised, edition of this rare pioneer work, published in London in 1692. This little volume of only 231 pages is very fascinating because of some of its quaint and confusing definitions, its unusual spelling, punctuation, and capitalization, and especially the light it throws on mathematical progress in the almost three centuries since it was given to the scientific world.

We find both "Mathematicks" and "Arithmetick" commonly used. Sometimes they are personified, either as masculine or feminine. Plane geometry is always "Plain Geometry," a characterization which geometry students today may sometimes wish were true! The use of italics is frequent.

A brief description of this old volume and samples of a few of its definitions may be of interest to mathematics teachers and to student members of mathematics clubs today.

Little is known of the author, Joseph Moxon, except the information given on the title page, reproduced on this page. The dedication of the revised edition, signed by James Moxon, says that "it was above twelve years since this Book was first Compos'd by my Father," then deceased.

Mathematicks made Easie:
Or, a Mathematical
DICTIONARY,
EXPLAINING
The Terms of Art, and Difficult Phrases used in *Arithmetick*, *Geometry*, *Astronomy*, *Astrology*, and other *Mathematical Sciences*.

Wherein the true Meaning of the word is Rendred, the Nature of Things signified Discussed, and (where Need requires) Illustrated with apt Figures and Diagrams.

With an *APPENDIX*, exactly containing the Quantities of all sorts of Weights and Measures: The Characters and Meaning of the Marks, Symbols, or Abreviations commonly used in *Algebra*. And sundry other Observables.

By *Joseph Moxon*, a Member of the Royal Society, and *Hydrographer* to the King's most Excellent Majesty.

The Second Edition, Corrected and much Enlarged by *Ben. Coley*, Teacher of the *Mathematicks* in *Baldwins Gardens*.

LONDON,
Printed for *J. Moxon*, at the Sign of *Atlas* in *Warwick Lane*. M. DC XCII.

The dedication reads as follows:

To the Honourable, CHRISTOPHER SEATON, Brother to the Right Honourable GEORGE Early of Winton, &c. This small (but useful) Treatise, is most humbly DEDICATED.

Sir, THE Excellency of your *Knowledge* in all kinds of Learning in General, and in Particular, that of the Mathematicks, hath embolden'd

me to the Presumption of Affixing your Worthy Name, as *Patron* to this Work; It was above twelve years since this Book was first Compos'd by my Father, and then Dedicated to a Noble *Patron* of this Nation; who are both Deceased: And now this Second Edition being Enlarg'd, Corrected, and Improv'd; I hope Sir! you will not impute it as a Crime in me, that I take this Boldness (being incouraged thereunto by your (constant Love and Favour)* to Shelter it for Protection under your Worthy Name. I hope, I need not doubt of your continued Friendship and Incouragement.

The Design of this *Treatise*, is to Inform Young Students in those hard *Terms* of *Art* used in the *Mathematicks*, and those *Laudable Arts* appertaining thereunto; and it may be Reasonably hoped, that it will be serviceable also to many Persons that have already made Considerable Progress in those Studies: which in themselves are not only Difficult, and therefore require a Tutor, but they are unlimited; for no man hitherto did ever arrive to a *ne plus ultra* therein.

I need not tell you, Sir! that the *Mathematical Arts* are grounded only upon those two main Pillars, *viz. Arithmetick* and *Geometry*, and thence are Deduced, and Demonstrated, those we call *Mix'd Mathematicks*, As *Astronomy, Navigation, Surveying, Dyalling, Optics, Fortification*, and many other curious *Arts;* Now since all the World cannot but freely akknowledge, not only their Excellency but Utility, therefore they are no less worthy of the Study and Industrious Acquirements of all true Lovers thereof, which will be Advantagious, not only to themselves, but to the Publick, as might be copiously Illustrated.

But worthy Sir! not to give you farther trouble, I humbly crave your Generous Acceptation of this *small Treatise*, which if I obtain, I shall then conclude it a Sufficient Protection, both of this Work, and my self, from all *Ill-affected*, and *Carping Criticks*, and may boldly adventure this *little Bark* into the wide Ocean of the World, without the least danger of *Shipwrack*, and *Ruine*, and thus I shall ever remain Honourable Sir! your most humble and obliged Servant, *James Moxon.*

This dedication, signed by Joseph Moxon's son, is followed by an eight-page section "To the Reader," all printed in italics. It reads in part as follows:

To Expatiate in Encomiums on the Mathematicks, were to Gild Gold; an Undertaking vain and impertinent. This sort of Learning has sufficiently justifi'd its Excellency, both from the Certainty of its Principles, and Usefulness of its Practice, to all Ingenious men. . . .

I have taken the pains to Collect and Explain the Hard Words, Difficult Terms, and Abstruse Phrases used by Authors, in all the several Mathematical Sciences, and Branches depending thereupon, than which, nothing has hitherto more discouraged hopeful Tyro's from Proficiency. But here is offered them a Key, that will open to them the Choicest Treasures of Urania, and her Sublunary Handmaids. . . .

If any Errors have been committed in the Interpretation let Imbecility of Judgment, or Defect of Memory be pardoned, but zealous good will for propagating of Arts be encouraged. which was never more cordially designed than in this Work: But the performance is submitted to the Charity of thy Censure.

Now in this Second Edition, the first Author being deceased, here is by another Hand (according to the Author's Request in his Lifetime) all other useful words Added, as could at thought off, that were Omitted in the first Impression, to make it the more useful and compleat for the young Mathematician's use. Vale.

There follows another eight-page section, "Of the Mathematicks in General, &c," the first three paragraphs of which read as follows:

The *Mathematicks* are Sciences, which being founded upon certain *Innate Principles* do teach Rightly and Easily to Explicate *Occult Properties*, and Intrivate Reasons of *Quantity*.

Or, the *Mathematicks* are Sciences of Quantity, the Foundation of other Arts, preparing the Learner for other parts of *Philosophy*.

The *Mathematicks* receive their Denomination from the *Greek* word *Mathesis*, which signifies *Discipline* and *Doctrine*, and doth not only signifie to Learn and Understand, but also to call to Remembrance the Knowledge of those things which are imprinted in the Mind, that is, a Remembrance raised from *Appearances;* as *Pythagoras* interprets it; or form'd by Knowledge it self, therefore the *Mathematicks* is such a Doctrine, whose Knowledge brings us to the Remembrance of those Principles which are imprinted in our Minds.

A little later, the author goes on to differentiate the principal branches of mathematicks as follows:

The Parts of the Mathematicks (at least those of most general use) are these which follow, *viz. Arithmetick* and *Geometry*, and these two Parts only are called *Pure Mathematicks*, from whence are deduced many other Branches called *Mixt Mathematicks*, As *Astronomy, Geography, Algebra, Trigonometry, Perspective*, which contains *Opticks, Catoptrics, Dioptrics, Musick*, viz. *Harmony* (or the Proportion of Sounds) *Chronology, Mechanicks, Staticks*, (or the Art of Ballancing) *Architecture, Fortification, Navigation, Surveying, Dyalling*, &c. To which may be added also the Art of *Astrology*, which is

* The three parentheses are in the text, as given here. Probably the second one, before "constant," is a printer's or proofreader's error.

grounded on, and depends upon *Astronomy; Astronomy* the Mother, and *Astrology* is the Daughter, according to *Kepler*.

It is noteworthy that Moxon considers arithmetic and geometry the only types of "Pure Mathematicks," while algebra, trigonometry, and many phases of applied mathematics are grouped together as "Mixt Mathematicks." The author goes on for several pages "to Inform the young Student what each of these Branches do contain," closing with the following observation:

> And thus you have a brief Account of the Excellent Use of the several Parts of the Mathematicks, which affords as much Delight and Satisfaction to the Industrious Student therein, as all the *Liberal Sciences* taken together, and the *Study* thereof is no less Profitable than Commendable.

The final preliminary section consists of eight pages of fifty-five geometrical definitions, some of which are repeated, sometimes in different and even inconsistent form, in the body of the work. A few of these definitions may be given as examples:

1. A *Point* is no Quantity, but said to be the Bound of a Line.
2. A *Right Line*, is that which lies even between 2 bounded Points.
5. A *Plain Surface*, is that which lies even between its two Bounding Lines, either as to Length, or Breadth, or both.
9. An *Obtuse* or Blunt Angle, is that which is Bigger than a Right.
12. A *Curv'd Lin'd Angle*, is made of two Crooked Lines, a mixt Angle is made by a crooked or arched Line and a streight one.
13. *Parallel* Lines, are those that lean not at all.
18. A *Scalene* Triangle, has no Side Equal.
20. An *Acute Angled* Triangle, has all his Angles Acute.
48. A *Cube* (or *Dye*) is a Solid Figure, contain'd under Six Equal Squares.

The body of the work (pp. 1–181) consists of some 900 definitions, arranged alphabetically, of a wide variety of terms. Somewhat less than half of these terms (including the fifty-five on geometry indicated above) refer to the two fields of "Pure Mathematicks"—arithmetic and geometry; about 40 per cent concern astronomy and its "daughter," astrology, while almost 10 per cent are scattered among navigation, surveying, dyalling, optics, fortifications, geography, architecture, and other fields.

Some of these terms are defined briefly in one or two lines. Others include discussions covering a paragraph or a page or more, as interpreted by Joseph Moxon, Fellow of the Royal Society, or by Henry Coley, who was responsible for the edition under consideration in this article. Incidentally, the final page of the volume contains a list of the various courses in mathematics and related sciences (chiefly as listed above) as "professed and taught by HENRY COLEY, Philomath, at his House in Baldwin Court over against the Old Hole in the Wall in Baldwin Gardens, near Grays-Inn-Lane."

A few samples of the definitions, as gathered in this "first mathematical dictionary ever published in any language," will illustrate the style of the volume and some of the concepts of mathematics and related sciences prevalent almost three centuries ago:

Addition is one of the five vulgar Rules of Arithmetick; and is no more than a putting together, or collection of two or more numbers into one.

Algebra, is an Arabick word, and signifies an abstruse sort of Arithmetick, the Art of Equation, or a certain Rule for the finding out the hidden powers of numbers, as well absolute as respective.

Area, in Geometry, the plain superficies comprehended between the sides of any Figure, plain or circular.

Axiom, A common Sentence, Principle, or Ground of any Art, generally taken for granted.

Azores are Islands in the North Latitude of 40 degrees, said by some to be situated at the true Western Meridian, whence they would Compute the Longitude of all places from hence, rather than the Fortunate Islands.

Bissextile, Leap-Year, When once in four years a whole day is added, to make up the odd six hours, which the course of the Sun yearly exceeds 365 days, being inserted, or put in next after the 24th of February.

Centesms, or Hundred parts of any Integer, whence Decimal Arithmetick is derived.

Chiliades, are the common Tables of Logarithms, Characteristique is the first Figure in each Number towards the left hand, and is always a Unite less than the Number of Place, the Logarithm represents.

Chorography, Called also *Topography*, is a part of *Geography*, which delivers the description of particular Provinces or Kingdoms only.

Circle, is a plain Figure, contained in one line only; in the midst whereof there is a Point, from whence all Lines drawn to the Circumference are equal.

Cometes, There are several opinions, or rather offers concerning the Matter they consist of; but the most vulgar opinion is, that they are Earthy vapours, having gross parts strongly compacted, which being drawn up sometimes even above the Orb of the Moon, are there set on fire, and continue flaming for some time; and then, the matter being consumed, disappear. They are generally thought to be as Beacons fired, to foreshow the approach of Divine vengeance, in Sword, Plague, Famine, or some other lamentable Calamity.

Compass, An Instrument Sea-men use to Steer the Ship by, much more easie to be understood by a Sight, than the best description.

Cone, A Body with a round flat Base, upon which every side of it is placed, so as it ends at the top, in a Point hanging directly over the Centre of the Base.

Cossick, The old word for Algebra, or Cosse.

Curved, Crooked, or a Body Hollowed.

Cypher or *Cyfer*, a Character used in Arithmetick, in form somewhat like the letter O, which alone or before (that is, towards the left hand of) any Numerical Figure, signifies nothing; but after another Figure, (that is, toward the right hand) increases that Figure ten times; if two Cyphers be placed after a Figure, they increase it an hundred times; if three, a thousand times; as 10, 100, 1000, &c. But in those Numbers called Decimals, they decrease towards the left hand in the like proportion.

Diagonal, In Geometry, a Straight Line drawn from one Angle of a Square to the opposite Angle.

Elements, The first Roots or Principles of Things, as Fire, Air, Earth, and Water, whereof all Bodies are composed. So Letters are called the first Elements of Learning; and Euclid's Fifteen Books of Geometry are so called, because without being acquainted with the Principles therein laid down, no Mathematical work can be undertaken and demonstrated.

Empyreum, The Heaven of Heavens, the Throne of God, Residence of Angels, and Eternal Mansions of Saints.

Exhalations, Vapours drawn up from the Earth or Sea into the Air, whereof are generated all kinds of Meteors, as Rain, Hail, Snow, Thunder, Lightning, Comets, Falling Stars, &c.

Extraction of Roots, The unravelling of a Power (or Number proposed as a Power) to find the Root.

Extream Reason, A Right Line is said to be divided according to Mean and Extream Reason, when-as the whole is to the greatest Segment, as the greatest Segment is to the lesser.

Figurate Number, Is a Number made by the multiplication of one Number or more by another.

Fraction, A broken Number expressing so many parts of a whole.

Focus, The Navel or 2 Centre points to draw the curv'd Figure, called an Elipsis, much like to an Oval Figure in form.

Geometrie, signifying no more in a strict sense, but *Measuring the Earth;* and therefore was anciently used for the two words last explained; but now it has been long appropriated to the most noble of all the Mathematical Sciences . . . Of which there is two Sorts, *Practical* and *Speculative;* of both which, Authors do treat very copiously in several Lanquages.

The Golden Rule, The Rule of Three is so intituled, by way of Excellency in Arithmetick, teaching from three Numbers known, to find out a Fourth unknown.

Hyperbola, A curved Figure used in solid Geometry. See Mr. Anderson's *Stereometrical Propositions in English*, to be had at the Black Raven, near to St. Andrews Church in Holbron.

Impure Logarithm, Or a *Defective Logarithm*, or the Logarithm of less than Unity.

Indivisible, A Term in Speculative Geometry, shews the thing cannot be divided.

Isosceles, A triangular Figure in Geometry, having two Sides only equal, but the third bigger or lesser than either of them.

A Line, is a Length void of Breadth, or as others define it, the Flux of a Point conceived to move from place to place, . . . As the Point, so the Line which we draw, is the Sign of that which we Conceive in mind: for if the Point which we Conceive be moved, and leaveth an Imaginary Tract, that shall be a Line; *Long*, by reason of the motion, but not Broad, because the Point from which it procedeth, is void of all Extension.

Logarithms, Are Artificial Numbers invented by Arithmeticians, to the end that being put in the place of Natural Numbers, they may be fit to manifest what Progressive difference there is in them: For they always keep in themselves the same Progression Arithmetical, as those in whose stead they are Constituted, do Progression Geometrical.

Luna, The Latin name for the *Moon*, the lower or next to Earth of the seven Planets, Feminine, and Nocturnal: For that borrowing her Light from the *Sun*, she excels in passive Qualities and Moisture: And therefore as the *Sun* presides over the Heart, Spirits, and Blood, so the *Moon* over the Brain, Bowels, and Phlegm. . . . There have not been wanting some, both Ancient and Modern Philosophers, that have conceited the Body of the *Moon* to be Inhabitable; but this seems repugnant to Divinity.

Magick, A good and innocent Science, teaching

the knowledge and mutual application of Actives to Passives, thereby performing many excellent and wonderful works. Such were the three *Magi*, or Wise men that came out of the East to worship our *Saviour*. But afterwards the Study being depraved by the Arabians, and fill'd with many Superstitious Vanities, the word came to be taken in an ill sense, for Conjuration, or some such wicked Art, that by confederacy with, and the assistance of the Devil, does either truly do, or rather in a jugling deceitful way seem to perform some Miraculous Operations, or above the ordinary attainment of Humane Nature. In which *Pliny* represents *Zoroaster* as the first Grand Master.

Marine, Anything appertaining to the Sea, and from thence *Mariner*, a Sailor or Master of the Ship, who ought to be skilled in the Mathematicks.

Mathematicks, The word *Mathesis* being originally Greek, signifies Learning or Discipline. And the Ancients concluding only those Sciences, which are founded upon Certainty, and proceed by Demonstration worthy of that name, called them, *Mathematicks*. They contain in general, Geometry, Arithmetick, Astronomy, and Musick; but indeed comprehend many other most excellent and useful Branches of Learning; . . . Of the Necessity, Profit, Extent, Life, Dignity, and Excellency of all which, see John Dee's most profound and incomparable Preface to *Euclid's Elements*.

Mechanicks, Are those Operations which are dispatch'd as well by the labour of the Hands, as of the Brain. . . . So such Workmen themselves are often called *Mechanicks:* A word ignorantly used by the Vulgar, in contempt, whereas there are scarce any Faculties more necessary to Humane Life.

Meteors, Are imperfect mixtures of the Elements drawn up by the Sun and Stars, and there hapning into several Forms. As sometimes where the matter is extraordinary, into Comets, blazing Stars, strange Appearances in the Air, Ignes fatui, &c, but commonly into Hail, Snow, and Hoary Frost, which are soon resolved again into the form of their Elements.

Milky-way, Is the only real Circle in Heaven; for in a clear night 'tis always conspicuous like a Swadling-band infolding the Constellations. . . . It appears with a kind of whiteness like Milk, but the cause thereof was much disputed, till our modern Telescopes convinc'd us, that 'tis only a heap of Stars.

Minute, Is the sixtieth part of a whole Degree, or Hour: So that every hour or degree of the Equator or Zodiack is divided into 60 Minutes, every Minute into 60 Seconds, each Second into 60 Thirds, and so to 10ths, or further, if you think you can carry your Conception so far.

Numeration, the first of the five Vulgar Rules of Arithmetick, teaching to read truly any Sum or Number, or write it down aptly when 'tis Propos'd.

Orbit, Is properly the Tract left by a Wheel in the Road; hence Astronomers use the word to signifie the way or course of the Sun, (peculiarly called the Ecliptick) as also of any other Planet moving on according to the Circle of its Latitude; nor is this without reason, for the transient Planet leaves behind him I know not what impression or footsteps of his Virtue and Quality, as Snails and other Insects leave marks of their tabifick quality in the path they pass over.

Orthogon, A Right Angle, or a Geometrical Figure which consists of Right Angles, and has an Equality of all its parts.

Orygon, Is a Geometrical Triangular Figure, having necessarily all its three Angles acute, in which it differs from an *Amblygon*, that must have but two acute, and the third obtuse.

Parabola, Is a Figure in Geometry, or Area, circumscribed by two Lines, one Right, and the other Crooked; The Right Line being an Axis for the crooked to move about upon, of which, the great *Archimedes* wrote much, and first of all, with many Arguments and Demonstrations attempted to Square it, and that not without success.

Parallelopleura, Are imperfect parallelograms and Irregular Correspondencies of the Angles or Sides whereby they are constituted.

Periphery, A Greek word signifying That Circular Line which goes about and infolds the whole Area or Content of a Circular Figure: the proportion of which to its Diameter is as 22 to Seven: The word is likewise sometimes taken for the whole Superficies of the Earth, or for the Convex and outermost part of any Celestial Sphere.

Plain, Amongst Geometricians, is a flat Superficies; that is, a Superficies that lies equal between its Lines, whereby 'tis opposed to a Spherical Body, and a Circle, which are circumscribed with a crooked Line, or Superficies.

A Point, is the smallest part of Quantity, or that Extream which can be divided into no further parts. The same (in a manner) in Quantity, as a *Unit* in Number, an *Instant* in Time, or a *Sound* in Musick.

Ponderous, Those Planets are said to be so that move leisurely and slowly, like a man under a burden; as *Saturn*, *Jupiter*, and *Mars*, who never by their Diurnal Motion can reach one whole Degree.

Prime Figure, Is that which cannot be divided into any other Figures more simple than it self: as a Triangle in Plains, the Piramis* in Solids; for all Plains are made of the first, all

* Moxon uses the word "Piramis," but whether this is a printer's error for "Piramid," or possibly "Piramids," may only be conjectured. Or perhaps the author was not as careful as he should have been to be consistent. Note that in the somewhat confusing definition of the word "Pyramid" immediately following, the modern spelling of the word is used consistently throughout.

Bodies or Solids compounded of the Second.

Pyramid, [so called from the Greek word *Pyr*, signifying Fire, because a flame ends Taper in a Point like this Figure] so a solid Figure contained under diverse Plains, which meet or Terminate, at one and the same point being drawn from another Plain which is for the Base of the Pyramid; and here note that every Body whether Opacous or Luminous transmits its shadow or Light after a Pyramidal Form. Only the Luminous Body casts it so, that the Vertex (Top) of the Pyramid form'd is in that point of the Luminous Body from whence the Light proceeds, and the Basis on the Superficies of the Body Inlightned, but an Opacous Body casts its shadow contrarywise, the Basis of the Pyramid by it made, being in the Superficies of the body making the shadow, and the *Vertex* in a point of any body to which the shadow is extended.

Quantity, Is the proper and largest object of Geometrical Arts, not as 'tis an accident adhering to matter, and endued with sensible qualities.

Right Line, Is that which is equally distended between its points; that is, hath nothing of turning or bending any way, and consequently of all Lines is in its nature the shortest.

Saturn, The slowest of the Planets, and most remote from the Earth, his Sphere being immediately next under the Heaven of the Fixed Stars, by reason of which distance, he seems to us the least, though in truth greater than any of all the Seven, except the *Sun* and Jupiter: Of a leaden, whitish, somewhat obscurist colour; by nature cold and dry, and so inimical to the nature of Man and all Creatures, that he is generally counted the Greater Infortune.

Serpentine Line, A crooked winding Line, that incloses it self continually, as a Serpent wraps her self up in folds. See *Spiral*.

Sol, The King of the Planets, Fountain of Light, and Eye of the World, by nature hot and dry; a Fortune by Aspect, but Infortune by Body; he has many names given him, both by Poets and other Authors, as Sol, Titan, Helios, Apollo, Paean, Diespiter, &c.

Spiral Line, A Tortuous or crooked Line, unequally distant from the midst of the space, howsoever inclosed, which seems to be almost a Circle, only it does not meet, and like that, run again into it self, but keeps on at a proportionate distance or deviation, like the Coyling of Ropes, or the folding of a Serpent, when she lies close in so many Turns with her Body, and is therefore sometimes called a Serpentine Line.

Squaring the Circle, Is a contriving to any Circle a Square equal thereunto, and exactly correspondent: A thing that has puzled the ablest Mathematicians, being in truth, to find out the Area of some Square that shall be exactly equal to the Area of some Circle, so that the Area's of both Figures shall be alike capacious. This, though the great *Archimedes*, and others, have not exactly done, yet they have come near enough for any use, and taught those things, which, if fully understood and pursued, the Circle may come to be Squared: For if they have Squared a Parabola, (which is an Area intercepted between two Lines, one Right, and the other Arcular, or bowing) why should not the Circle it self, which consists of two Parabola's be as well Squared?

Subtraction, A common Rule of Arithmetick, teaching, how artificially to take a lesser Number out of a greater, and find what remains over-plus of the greater, when the value of the smaller is subtracted, that is, withdrawn, or taken away from it.

Systeme, A Greek word whereby is intended the general Constitution, Fabrick and Harmony of the Universe, or an orderly Representation thereof according to some noted Hypothesis, wherein the Celestial Bodies are so disposed amongst themselves, and in respect of the Earth; as their Scituation, Order, Motions, and Passions, may in such an Authors opinion best answer Appearances and Philosophical Demonstrations; To this purpose the Ancients agreed, that the Globe of the Earth and Sea should be the Centre of the World, about which was diffused the Air, and round that the more light Element of Fire; Then the Orb of the Moon, and so Mercury. . . . But of late, *Copernicus* (a most ingenious Astronomer) the better to Solve the Appearance Invented, or rather revived, for it was many Ages before started by some old Phylosophers, but now buried and forgot) he, I say, revived another Hypothesis, which constitutes the Sun Immovable in the Centre of the World, next the Orb of Mercury. . . . Since this, the Noble *Tycho Brahe* proposed his Hypothesis, making the Earth the Centre of both the Luminaries, and the Orb of fixed Stars, but the Sun the Centre of the other four Planets; whom he counts all Eccenttrick to the Earth, and constantly moving in the fluid Ether about the Sun, &c. All these Systemes have found their Disciples, and Propugnators; if any have a mind to be better Instructed in all or any of them, there are Sphears extant fitted for every of them, and shewing all the Motions and Affections of the Celestial Bodies truly, according to the intention of each Hypothesis.

Telescope, a large Optick Instrument, invented by *Galilaeus*, for observing the Celestial-Bodies, whereby several new Phaenomena have been discovered and great Improvements made in Astronomy.

Theorem, A Greek word signifying a Proposition, which requireth the searching out, and demonstration of some Property or Passion of some Figure, wherein is only Speculation and Contemplation of mind, without doing or working of any thing; whereby it differs from a Problem, in which something is always to be done.

Hints for upgrading mathematics instruction—new or old?

by Sister Mary Constantia S.C.L., Ward High School, Kansas City, Kansas

Current textbook writers are placing much emphasis on teaching children how to perform the fundamental operations—addition, subtraction, multiplication, and division—in bases other than ten. This is advocated (we believe rightly so) because it enables the learner to really understand the concepts involved, and he is not so apt to compute mechanically according to a memorized rule. If the child has developed the habit of mechanical manipulation of numbers, the change to bases other than ten is perhaps the best approach to use in giving him proper concepts.

It is interesting to note, however, that as early as 1836 Augustus De Morgan (1806–1871) had published a paper in which he also recommended using different bases to aid in forming correct concepts in the fundamental operations. The *Encyclopaedia Britannica*, 11th edition, says of De Morgan, "As a teacher, he was unrivalled."

Concerning change of bases, De Morgan states, "The student should accustom himself to work questions in different bases of numeration, which will give him a clearer insight into the nature of mathematical processes than he could obtain by any other method."

De Morgan wrote prolifically in mathematics and also in logic. He is the author of the volume *Mathematics*, one of a series of books on different subjects in the *Library of Useful Knowledge* sponsored by the Society for the Diffusion of Knowledge. Three of his numerous papers are collected in this volume. The title of the first paper is, "On the Study and Difficulties of Mathematics." Chapter II, entitled "Arithmetical Notation," gives what I consider an excellent presentation of a lesson in *base* and *place* in numeration. In what follows I shall quote at length from his presentation.

De Morgan prefaces his lesson with the observation:

> A knowledge of our method of reckoning and of writing down numbers is taught so early that the method by which they begun is hardly recollected. Few, therefore reflect upon the commencement of arithmetic or upon the simplicity or elegance with which calculations are conducted. We find the method of reckoning by tens in our hands, we hardly know how, and we conclude, so natural and obvious does it seem, that it came with our language and is a natural part of it and that we are not much indebted to instruction for so simple a gift.

He points out, however, that numerals are arbitrary symbols used to represent the counting numbers whose names were also chosen arbitrarily. He says there are nine symbols chosen, whereby we can represent the first nine counting numbers and one sign for zero or nothing. But he explains, "To represent *ten* we do not use a new sign but combine two of the others and denote it by 10, eleven by 11, and so on."

Then he asks, "But why was *ten* chosen as the limit of our separate symbols—why not *nine* or *eight* or *eleven*?" He gives the answer:

> If we recollect how apt we are to count on our fingers we shall be at no loss to see the reason. We can imagine our system of numeration formed thus: A man proceeds to count a number, and to help he puts his finger on the table for each one he counts. He can thus go as far as ten, after which he must begin again and by reckoning on the fingers a second time he will have counted twenty and so on, but this is not enough; he must reckon the number of times he has done this, and as by counting on his fingers he has divided the things which he is

counting in lots of ten each, he may consider each lot as a unit of its kind, just as we say a number of sheep is *one* flock, twenty shillings are *one* pound. Call each lot *ten*. In this way he can count a ten of tens which we may call *one* hundred, a ten of hundreds, or *one* thousand and so on.

De Morgan gives a concrete example to illustrate his lesson:

The process of reckoning would then be as follows:—Suppose to choose an example, a number of faggots is to be counted. They are first tied up in bundles of ten, until there are less than ten left. Suppose there are seven left over. We then count the bundles of ten as we counted the single faggots and tie them up also by tens forming new bundles of one hundred each with some bundles of ten remaining. Let these be six in number. We then tie up the hundreds in tens, making bundles of thousands, and find there are five bundles of hundreds remaining. Suppose in attempting to tie the thousands by ten, we find there are not as many as ten but only four, the number of faggots is then: 4 thousands, 5 hundreds, 6 tens, and 7.

Again De Morgan stimulates the reader to think by asking, "How shall we represent this number in a short and convenient manner?"

He explains the manner of doing it is a matter of choice. He suggests that we might distinguish the tens by marking their numbers with a single accent, the hundreds with two accents, and the thousands with three, leaving the units with no accent. Having made this suggestion, he points out that the order in which we wrote the accented symbols would be immaterial. Hence the number of faggots counted above could be written: 7 6′ 5″ 4‴, 6′ 5″ 7 4‴, 6′ 4‴ 5″ 7, or 4‴ 5″ 6′ 7 (there being actually 24 ways the symbols could be combined to represent the number). But he concludes, one certain order is sufficient, and the most natural way is to place them in order of magnitude, putting the largest collection first. Then the number of faggots is represented as 4‴ 5″ 6′ 7.

He then observes, if we write the numerals thus, it will soon be evident that the accents are unnecessary. The singly accented symbol will always be second to the right and so on, and the *place* of each number will determine what accent is to be understood as belonging to it. Thus, we may consider each figure as deriving a value from the place where it stands.

Here De Morgan anticipates a difficulty and remarks, "Here a difficulty occurs. How are we to represent the numbers 3‴ 3′ and 4‴ 2′ 7 without accents? If we write them 33 and 427, they will be mistaken for 3′ 3 and 4″ 2′ 7."

He resolves this difficulty by explaining we can place cyphers so as to bring each numeral into the *place* allotted to the collection which it represents. Thus since the trebly accented symbols or thousands are in the fourth place from the right, and the singly accented symbols in the second place, the numeral 3‴ 3′ can be written 3030 and 4‴ 2′ 7 can be written 4027.

It is apparent De Morgan considered zero only as a place holder and not as a number in its own right, for he says, "The cypher which plays so important a part in arithmetic that it was anciently called the *art of cypher* or *cyphering*, does not stand for any number in itself, but is merely employed like blank types in printing to keep other signs in those places they must occupy in order to be read rightly. "

Again De Morgan introduces a new idea by asking questions. He asks: "What would have been the case if instead of ten fingers men had more or less? For example what signs would represent 4567 faggots if man had nine fingers instead of ten?"

He answers that the method would be the same—man would count by nines. There would not be a single symbol for nine, but instead it would be represented by 1′ 0. To make this idea clear he shows how the original group of 4567 (ten base) faggots could be counted by nines and tied in bundles of nine, with four single faggots remaining. The nine bundles would again be tied in larger bundles, nine of nines (81), with 3 bundles remaining. When the nine of nines are in turn tied in bundles of nine, there would be six such bundles (729), and two nine of nines (81) remaining. The process would end there.

A bundle of 9 single faggots or a *nine* would then be represented by 1′, a *nine of nines* by 1″, and a *nine of nine of nines* by 1‴. The numeral which represented the number in the ten base would now be changed to 6‴ 2″ 3′ 4 to represent that number in the nine base.

To avoid confusion of bases in his illustrations, De Morgan does not use the accents for the common base ten, but uses them whenever numerals in bases other than ten are employed. Thus, he exhibits side by side the following numerals in base ten and in base nine:

Counting by tens	*Counting by nines*
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	1′ 0
10	1′ 1
11	1′ 2
12	1′ 3
13	1′ 4
14	1′ 5
15	1′ 6
16	1′ 7
17	1′ 8
18	2′ 0
19	2′ 1
20	2′ 2
30	3′ 3
40	4′ 4
50	5′ 5
60	6′ 6
70	7′ 7
80	8′ 8
90	1″ 1′ 0
100	1″ 2′ 1

De Morgan illustrates how to change from a ten base to a nine base.

We will now write in the common way in the tens' system the process we went through in order to find out how to represent the number 4567 in that of nines thus:

9)4567

9)507 Rem. 4 — Representation required:

9)56 Rem. 3

9)6 Rem. 2 — 6‴2″3′4

0 Rem. 6

Then De Morgan makes the following statement: "The principles involved in computation in a numeration system which employs *place* values are the same irrespective of the *base* used." To make this clear he gives examples in each of the four fundamental operations, solved first in numerals of base ten, and then the same numbers in numerals of base nine are used. These illustrations follow. (Note that the accents are not used in the base ten numerals, but they are employed in base nine. Notice also in the division problem that, where more than three accents would be needed, he changes to Roman numerals for superscripts. However, he drops the accents in the actual computation.) Before setting down his examples, he cautions the reader, "There is this difference that in the common base the *tens* must be carried, in the second we carry *nines.*"

Examples:

Addition:

Base *ten*	Base *nine*
6 3 6	7″ 7′ 6
9 8 7	1‴ 3″ 1′ 6
4 0 3	4″ 8′ 7
2 0 2 6	2‴ 7″ 0′ 1

Subtraction:

1 3 8 4	1‴ 8″ 0′ 7
7 9 7	1‴ 0″ 7′ 5
5 8 7	7″ 2′ 2

Multiplication:

2 9 7	3″ 6′ 0
1 3 6	1″ 6′ 1
1 7 8 2	3 6 0
8 9 1	2 4 0 0
2 9 7	3 6 0
4 0 3 9 2	6^{iv} 1^{iii} 3^{ii} 6^{i} 0

Division:

6 3 3) 7 9 1 2 5 (1 2 5

6 3 3

1 5 8 2

1 2 6 6

3 1 6 5

3 1 6 5

0

$7''\ 7'\ 3\ )\ 1^{v}\ 3^{iv}\ 0^{iii}\ 4^{ii}\ 7^{i}\ 6\ (\ 1''\ 4'\ 8$

$$\begin{array}{r} 7\ \ 7\ \ 3 \\ \hline 4\ \ 2\ \ 1\ \ 7 \\ 3\ \ 4\ \ 2\ \ 3 \\ \hline 6\ \ 8\ \ 4\ \ 6 \\ 6\ \ 8\ \ 4\ \ 6 \\ \hline 0 \end{array}$$

Following these examples, we find the advice to the student to accustom himself to working in different systems of numeration, in order to get a clearer concept of the processes of arithmetic. He also explains that when a base larger than ten is used, we shall need some new symbols for numbers greater than nine and less than the base. He says, for instance, in the duodecimal system the base *twelve* will be represented by 10 and we will need to agree on single distinct signs for *ten* and *eleven*. He suggests that we reverse the symbols for nine and six, thus ℓ, ∂, and use ℓ for *eleven* and ∂ for *ten*. The numerals for the numbers one through twelve would then be

1, 2, 3, 4, 5, 6, 7, 8, 9, ∂, ℓ, 10.

The lesson on *base* and *place* is but a single example of the many ideas expressed in De Morgan's writings that have a distinctly "modern flavor" as we examine them today.

To cite only a few others: Is he not in keeping with today's thinking when he admonishes the teacher to proceed slowly with simple examples, so that the student will be led to discover the concepts involved, instead of being presented with a *rule* for working the problem? This is surely our goal for our own students.

De Morgan warns, "He [the student] may, and indeed will, believe a rule on the authority of the instructor, but this disposition is to be checked. He must be told whatever is not gained by his own thought is not gained to any purpose." He also states, "Merely showing a student a rule by which he is to work, and then comparing his answer with a key for the preceptor's use only is *not* teaching arithmetic."

Like many of our leading mathematicians today, De Morgan realized that teachers must first be educated properly if we are to expect any improvement in students' mastering of the subject matter. This is shown by the following quotation taken from the preface to "Arithmetic and Algebra," the second paper in the volume *Mathematics* referred to above: "There are two classes who might benefit by this paper, viz. teachers of the elements,* who have hitherto confined their pupils to working of rules, without demonstration, and students, who, having acquired some knowledge under this system, find further progress checked by the insufficiency of their previous methods and attainments." To the latter class he says his paper presents "the part which has been omitted in their mathematical education."

Further on in the Preface he expresses the hope that "the persual of the opinions of an individual offered simply as such, may excite many to become enquirers, who would have otherwise been workers of rules and dogmas." He states these enquirers may not ultimately agree with his opinions, but if he has stimulated enquiry he feels he has accomplished his purpose.

Did De Morgan at this early time envision some sort of plan or idea similar to The Mathematics Study Group, where trained mathematicians would sit down side by side with high school and elementary-school teachers to plan a better mathematics curriculum for the pupil as well as helps for the teacher in presenting this improved curriculum? He must certainly have thought some means possible. There is no doubt that he placed the responsibility for improvement squarely on the shoulders of the trained mathematicians. He rebukes them for having thus

* De Morgan says, "The elements of mathematics taught are usually contained in the sciences of arithmetic, algebra, geometry, and trigonometry." Hence his reference to "the elements" is not to Euclid's *Elements*.

far failed in their duty. This is shown in the final paragraph of the Preface:

The number of mathematical students, as it has been of late, would be much augmented if those who hold the highest rank in the science would condescend to give more effective assistance in clearing the elements of the difficulties they present. If anyone claiming that title, think my attempt obscure or erroneous, he must share the blame with me, since it is through his neglect that I have been enabled to avail myself of an opportunity to perform a task which I would have gladly confided to more skillful hands.

The Preface was signed: AUGUSTUS DE MORGAN, 90 Guilford Street, November 22, 1831.

Letter to the editor

Dear Editor:

In the February, 1960, issue of *The Mathematics Teacher* there appeared, on page 119, an interesting article that concerned the dissection of a square into parts that form an equilateral triangle. After inspecting the proof, I decided that it was incorrect and then proceeded to investigate and discover a correct proof.

Reference to page 129 of this magazine indicates that the author contends that two quadrilaterals, that he has named 3 and 3′, are congruent, because they are mutually equiangular, and a side of one is equal to the corresponding side of the other. This, of course, is false, since two corresponding sides were needed. The proof, therefore, fails at that point and becomes inconclusive.

The construction that I offer is the same as the original up to the determination of a certain point G. At this point, there is a slight difference in the procedure that will make it possible to prove the quadrilaterals 3 and 3′ congruent.

The accompanying diagram is lettered like the original.

There is given a square $ABCD$, with the length of each side denoted a. Let s denote the length of side of an equivalent equilateral triangle.

From this hypothesis it follows:

$$a^2=\frac{s^2\sqrt{3}}{4}.$$

By a compass and ruler construction (see page 122) a length equal to $\frac{1}{2}s$ is determined. The midpoints E and N on the sides of the given square are taken. Next take $EF=\frac{1}{2}s$, and $NG=\frac{1}{2}s$. Since it is known that $FN<\frac{1}{2}s$, there must be a point such as G on line EF.

Let $AF=k$. Take $CM=k$. Next draw GM, which will be of an unknown length. Extend GN to O so that $GO=s$. Now draw OX parallel to GM, meeting EF, extended, at P to form a triangle GOP.

Next take $FR=k$. Draw RS perpendicular to AQ.

This triangle GOP will now be shown to be equivalent to the square $ABCD$.

Consider: 2 and 2′.

$$BN=CN, \qquad GN=NO.$$

Obviously the quadrilaterals are mutually equiangular. Hence

$$2\cong 2' \qquad \therefore\ BQ=k.$$

Consider: 4 and 4′. These triangles are congruent, since two angles and the included side of one are equal respectively to two angles and the included side of the other. Hence

$$RS=\tfrac{1}{2}a, \qquad \text{and } FS=\tfrac{1}{2}s.$$

Consider: 3 and 3′.

$$FB=a-k, \qquad BR=k-(a-k)=2k-a$$

It follows:

$$BQ-RB=k-2k+a=a-k.$$

Hence

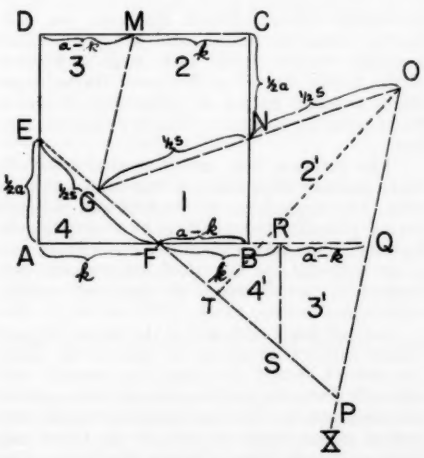
$$RQ=a-k. \quad \text{But } DM=a-k, \text{ also.}$$

Therefore:

$$RQ=DM$$

Also

$$RS=\tfrac{1}{2}a=DE.$$



(*Continued on page 269*)

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Basic College Mathematics, Jerome M. Sachs, Ruth B. Rasmussen, and William J. Purcell (Boston: Allyn and Bacon, Inc., 1960). Cloth, ix+344 pp., $6.95.

Slightly more than half of this text is devoted to restudy of arithmetic and elementary algebra. The opening chapter introduces some logical concepts and symbolism, and there is occasional use of these throughout the book,

A section on geometry is concerned mainly with mensuration of plane and solid figures. In the reviewer's judgment, this chapter falls behind the others in reaching the announced goal that it shall "integrate and expand, within a modern framework, the learning acquired in algebra and geometry." Two other chapters, of an optional nature, are concerned with an introduction to statistics and assorted topics.

The book seems to the reviewer to be planned particularly for the liberal arts student who does not plan to continue the study of mathematics, although there is no such explicit statement in the Preface. It should certainly be examined by those looking for a text for a semester course of this type, at an elementary level.

It is somewhat harder to place this text in a sequence of mathematics courses since the technical facility to be expected from its use would be somewhat short of that desired in intermediate college algebra. However, the well-known excessive failure rate in such courses certainly results partly from lack of mastery of the topics treated in this text. Hence, some might consider its use in a remedial course, to be followed by a more extensive course in algebra.

The problem lists are of moderate length, with answers supplied for the odd-numbered ones. The appearance is excellent and the style seems pleasing but lacking any considerable novelty. The authors seem to have written with some restraint, as though some revision were desired in the direction of "modern" mathematics, but not too much.

Several texts with much the same title and course content are already in use, so the following remark should not imply any special criticism: Can't we hope that some day this material will be *basic* in the high school or junior high school rather than in *college?—L. Clark Lay, Orange County State College, Fullerton, California.*

Cartesian Geometry of the Plane, E. M. Hartley (New York: Cambridge University Press, 1960). Cloth, xi+324 pp., $3.75.

The contents of this book are, in order: line, plane, parabola, parametric definition of curves, circle, ellipse and hyperbola, line pairs, general conic, polar co-ordinates, and a brief summarizing chapter entitled, "What Is a Conic?"

Parametric representation permeates the entire book. It is introduced and defined in Chapter 2 in connection with study of the line. It is utilized again in Chapter 3. It is presented explicitly and applied in detail in Chapter 4, and use is made of parameters thereafter as it is advantageous.

In contrast, and of possible disappointment to some teachers, is the late and very brief treatment of polar co-ordinates. No more is made of the relations between rectangular and polar co-ordinates than mere mention, and almost no practice is provided for converting from one system to the other.

Translation and rotation of axes are presented early, but as optional topics. Neither transformation is used in the text or exercises until late in the book, when the general second-degree equation is considered, not, however, as an optional topic.

Five methods usable for defining conics are presented in the final chapter, some of which have been used earlier as the various conics were taken up in detail. Advantages and deficiencies of each approach are summarized. For example, the focus-directrix definition does not yield a circle, and use of the sum or difference of distances from two fixed points does not lead to the general line pair.

In several places the progressive development of a mathematical idea is evident. To cite one example, development of the locus of a point equidistant from two fixed points is followed almost immediately by consideration of the locus of a point, such that the ratio of its distances from two fixed points is any constant.

Alternate developments by means of the calculus, especially where slope is involved, are given in some places. A set of exercises follows each major section or set of sections in each chapter, and a lengthy set of miscellaneous exercises is provided at the end of each chapter. Hints for some of the more challenging exercises and answers to many of the exercises are given in sections at the end of the book. There is also a section listing the important formulas. Throughout the book helpful amplifying dis-

cussions in smaller print supplement the basic presentation.

Proof of important theorems of plane geometry by co-ordinate methods receives major emphasis in the first two chapters. Later on, in discussing the relative popularity of algebraic versus synthetic methods for dealing with the conics, the author makes a statement that has been used in support of employing co-ordinate methods in the study of high school geometry. He says, ". . . in solving a problem we should look for an economical solution, using either algebraic or synthetic reasoning, or probably a judicious mixture of the two."

There are a few words and symbols that are disconcerting to those accustomed to mathematics texts written and published in the United States. (The book is printed in London, and it is assumed that the author received all or much of his education in Great Britain.) For instance, $P_1P_2^2$ symbolizes what we are accustomed to seeing as $\overline{P_1P_2}^2$ or $(P_1P_2)^2$. The word "join" is used as follows: "If any two points on a line are selected their join makes a constant angle with a fixed direction. . . ." Especially troubling is the fact that it is not clear from the context whether "join" refers to "segment" or to "line." Less bothersome, but possibly unfamiliar to some readers, is the word "gradient"; however, the meaning as being synonymous with that of "slope" would be inferred from context. A word appears to have been added to or omitted from that intended in the sentence, "The intercepts on the x-axis and the y-axis are the $p/\cos\ \alpha$ and $p/\sin\ \alpha$ . . . ," but this may be a manner of expression customarily used by the author.

On the whole, reading of and reference to this text for co-ordinate geometry will be most rewarding. Extensive attention to the cubic equation, the many properties of the conics not found in most similar texts, the inclusion of inversion and related topics in the study of the circle, and the many tantalizing exercises persuade me to recommend it for reference use by any teacher of a course in analytic geometry, or by anyone desiring to extend his knowledge of the subject, or even by one wishing to become acquainted with it. Indeed, use of this book as a text in a substantial course, in the opinion of the reviewer, would contribute greatly to the mathematical development of the students and provide a stimulating experience for the teacher. *—Clarence H. Heinke, Capital University, Columbus, Ohio.*

Intermediate Algebra, Francis J. Mueller (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960). Cloth, x+374 pp., $5.95.

The student who comes to college so badly prepared in mathematics that he needs a course in Intermediate Algebra would find this book a useful compendium of rules, procedures, and exercises. He might, by careful study of this book, develop reasonable skill in techniques. For an understanding of the subject as a whole, for an appreciation of it as a sensible part of his general education, he would have to look elsewhere. The opening sentence of the Preface calls Intermediate Algebra "a nebulous, ill-defined entity." After reading this book one remains inclined to agree with this description.

In general, the language of the book is a mixture of modern usage, ancient habit, and personal peculiarities. For instance, on page 18 we find, "subtract the lower numeral from the upper." On page 60 appears "divide both members equally." On page 67 we find "Translate the following statements . . . ," but most of what follows are not statements. On page 91 the author mentions the distributive axiom and then ignores it in the rule on page 92. On page 13 he says, "Ignore the signs," when the concept of absolute value could have been easily introduced. One is sure to come out of a study of pages 256–260 with a feeling that "solving a proportion" is somehow different from solving an equation. The word "inequality" is not even indexed. The symbols, $>$, $<$ occur a few times, but the subject of inequalities is not discussed or even suggested.

On page 4 the author lists "principles" instead of axioms. It is difficult to see what has been gained by this dodge except to emphasize again the smothering by the book of the structure of the subject.

The author mentions the number of rules and certainly does not exaggerate. There is a "rule" for almost everything. Specifically set out and labeled as "rules" are seventy-four different sets of statements. Most books undoubtedly have as many such statements. However, to label them all in boldface type as rules tends to create the feeling that here we have a set of mechanical dodges that work, a point of view whose dominance has been fought for years by teachers.

The author constantly uses nouns as adjectives and thereby produces some rather uncommon effects. For example, one finds the following: (The number indicates the page.)

sum numeral (3)
integer problems (76)
equation statement (81)
quotient factor (92)
expansion terms (96)
lowest terms fraction (121)
dividend fraction (122)
common denominator pairs (124)
equality statement (180)
second degree variable term (184)
specific coefficient values (188)
number line axes (207)
independent variable replacements (212)
ratio fractions (257)

There are others, but these may illustrate the point.

There are many exercises, and it is certainly true that the discussions and rules take one as close to a skill for "doing" the exercises as may be possible without real understanding. Real

understanding, for the college student who knows little mathematics to begin with, would be difficult to achieve from a study of this book.

It is quite unsuited for the secondary school. This statement is no criticism of the book, however, for the implication of the Preface is that it was written for college students.—*Jackson B. Adkins, The Phillips Exeter Academy, Exeter, New Hampshire.*

Modern College Algebra, Julian D. Mancell and Mario O. Gonzales (Boston: Allyn and Bacon, Inc., 1960). Cloth, xi+386 pp., $6.25.

The aim of the authors of this text is to provide an elementary course in algebra for college freshmen that is teachable and mathematically trustworthy. They have tried to avoid the error of making college algebra a prolonged drill on techniques by using an approach based on understanding. Considerable emphasis is given to the number systems of algebra. Brief developments of the natural numbers, rational fractions, and irrational numbers are followed by a listing of the postulates for the real numbers, with derivations and some of their properties. Complex numbers are defined by the ordered-pair approach with the basic operations defined so that the principle of permanence of basic properties is satisfied.

Analytic geometry is introduced primarily for graphical representation of relations and functions. It is not Euclidean, but utilizes the properties of affine geometry.

The word "modern" in the title is a bit misleading, for the major part of the content of the book, as well as its treatment, is, in fact, traditional. True, the associative, commutative, and distributive laws are called upon faithfully when operations with polynomials are presented. Yet as one looks through the lists of exercises, one sees in most cases the traditional type of drill. An example of this is seen where practice in proof by mathematical induction is given. Traditional work on summation of series is given. Double induction is not mentioned, nor are ways to circumvent it mentioned, yet the proof of $a^m a^n = a^{m+n}$ is "left as an exercise." In the section on radicals, certain statements, made without proof, are easily proved by induction and thus this basic principle could have been more thoroughly exploited to develop the structure of algebra.

There is no shortage of topics. Most of what goes into college algebras is there. I would characterize this textbook as traditional with a leaning toward modern.—*Lyman C. Peck, Ohio Wesleyan University, Delaware, Ohio.*

A Teaching Guide for Slide Rule Instruction, Maurice L. Hartung (Chicago 5, Illinois, Alhambra, California: Pickett & Eckel, Inc. 1960). Paper, 45 pp.

This teaching guide is mathematically correct and pedagogically sound. The author states in the introduction that many teachers have not had extensive experience in teaching the slide rule. This is one of the areas in colleges and universities in which no teaching specialty is made.

Teachers are looking for suggestions for effective ways of introducing the slide rule to their students. Other teachers who have taught the slide rule may be interested in comparing their methods with those that have been found to be effective by others. It is even helpful to find that some methods used may correspond. This teaching guide is written for students in the junior and senior high school courses, particularly those who have not had trigonometry, as the book has been presented in such a manner that the mechanics involved do not need the thorough explanation of logarithms.

The guide is organized in twelve sections. The instructor may use his prerogative in using all the sections and in the order given, or in choosing those appropriate to the maturity of, and time allotted for the individual classes. There is also some freedom of choice as to the order in which the various topics may be taken up.

The illustrations and pictures are excellent, and the language used in the explanation is clear. The illustrations of the graduations are clearly presented so that students able to read and understand scales can readily understand the slide rule graduations. In explaining that the primary graduations are closer together as one moves from left to right, the students are told that the scales are nonuniform. This does not explain the why, but without a thorough knowledge of logarithms or exponentiation, they wouldn't understand. Since the principal purpose of teaching the slide rule at an early age is its use, the underlying mathematical background can wait.

There is a short section on approximations which should be especially effective. This is very important, whether a student studies slide rule or not. One of the great faults that teachers find in their students is not in their errors but in having an unreasonable answer. This section on approximations is followed by one on scientific notation. If the order were reversed, possibly most students would never learn to approximate. The scientific notation is not only useful in the slide rule computations, but it has widespread use in the modern scientific world.

One of the sections deals with a "Powers-often" slide rule as a learning aid in teaching exponential notation and decimal point location. It is really an alternative approach to scientific notation through the use of a special learning aid. At the same time, it introduces the soundest approach to a purely mechanical way of locating the decimal point. If I were to make a suggestion regarding the organization of the sections, I would have this section precede the one on scientific notation.

In addition to the above-named sections, there are sections dealing with *A* and *B* scales, the *K* scale, *CI* scale, the *L.C.D.* scales, *S.T.* and *ST* scales, and the teaching of slide rule operation in terms of logarithms.

The author is modest in saying that if each

teacher who looks through this guide finds only one suggestion that will help him make the study of mathematics more interesting and effective for his students, the preparation of the guide will have been worth the effort.—*William B. Lantz, Mansfield Senior High School, Mansfield, Ohio.*

FILM

Language of Algebra, International Film Bureau (Chicago: International Film Bureau, 1960). 16 minutes, color. $165.

Viewing this film is a pleasant experience. The use of happy music, pleasing colors, animated experiences that are easily understood, and plenty of action with very little "heavy" discussion all help in making this a fast-moving period of time.

The little story begins with the idea of symbols—something that substitutes for things or ideas. Everyday experiences, such as the use of a policeman's star, direction arrows, and stop-and-go lights introduce this idea. This leads to the main animated story of the development of a subdivision and how quantitative portions of this story can be told by the use of symbols. Some interesting aspects presented are the abstract name by which the subdivision is known, its perimeter (sidewalk), the area (different lots), and developing and fencing in a square playground.

The film closes with some feeling for new types of symbols, such as holes in punched cards.

Some interesting sequences of development are presented. The perimeter is found by the equation $s=l+w+l+w$ or $s=2l+2w$ or $s=2(l+w)$. The viewer sees these different expressions developed without any difficulty, but there is no discussion of the mathematics nor of the physical significance of these changes in the formula. The parentheses are used in several instances without comments. One type of such use is for substitution of specific quantities in formula $A=lw$. After the substitution the formula becomes $A=200$ (300). Later, a formula for the amount of rope needed for some swings was developed as $R=n(2D-2d)+k$. It is then suggested that the 2 be "taken out" of the parentheses and the formula becomes $R=2n(D-d)+k$.

Subscripts are used as a means for designating the number of people in the houses, such as $N_A+N_B+N_C+N_D$. This idea of subscript had been presented earlier to designate the area of one of the four equal lots when the total area was known. This was given as $A_p=\frac{1}{4}A_R$. An interesting point which is made in the film is that the area remains constant but that the number of people changed or varied as years passed.

Throughout the presentation color cuing is used to indicate the specific item or items being discussed. This seems to be an effective technique to hold interest and direct attention.

A feeling for the need and use of symbols is developed by viewing this film. Little attempt is made to develop a feeling of preciseness in the language and operations of mathematics. The teacher wishing to bring out these points would need to do so by discussion of the formulas and solutions of exercises as presented in the film.

The film would probably be most effectively shown during the early period of algebraic experiences for the pupil. This might be in the seventh or eighth grades or possible early in the ninth grade.

I was misled by the title as to the scope to be covered in the film. The title might better convey the idea that this film introduces the use of some symbolism used in algebra.—*Francis R. Brown, Illinois State Normal University, Normal, Illinois.*

Letter to the editor

(*Continued from page* 265)

By construction, 3 and 3′ are evidently mutually equiangular. It follows that

$$3\cong 3' \quad \therefore \quad EG=SP, \qquad GM=PQ.$$

Now it can be seen that:

$$1+2+3+4=1+2'+3'+4'$$

or, the Square $ABCD$ is equivalent to the triangle GOP.

The area of $\Delta GOP=a^2$, being equivalent to the square.

Now consider ΔGOP:

$GP=s$, since $FS=\frac{1}{2}s$ and $GF+SP=\frac{1}{2}s$

$GO=s$, being made so.

It follows ΔGOP is isosceles. Let h be the altitude of ΔGOP from the vertex O. Then: area $\Delta GOP=\frac{1}{2}hs$. However, area

$$\Delta GOP=a^2=\frac{s^2\sqrt{3}}{4}.$$

Therefore,

$$\tfrac{1}{2}hs=\frac{s^2\sqrt{3}}{4} \quad \text{or} \quad h=\tfrac{1}{2}s\sqrt{3}.$$

Let $GT=m$. Then

$$s^2=m^2+h^2, \quad \text{or} \quad s^2=m^2+\frac{3s^2}{4}.$$

Hence $m=\frac{1}{2}s$. Since the altitude bisects the base of the triangle, it must be isosceles. Hence $OP=s$ and consequently ΔGOP is equilateral. In conclusion, it has been shown that the square $ABCD$ is equivalent to a triangle GOP that is equilateral.

ENOR E. LUNDIN
Medford, Massachusetts

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Rapid calculation for the teacher

by Brother U. Alfred, F.S.C., St. Mary's College, California

One of the interesting by-products of the teaching profession is the contrast between the self-image—what we conceive ourselves to be and what we regard as important in our work—and the image we leave in the minds of our students. For the two can be like Kipling's East and West! I have noticed this phenomenon when meeting former students of mine. What do they remember? Not the manner in which I solved difficult problems or threaded my way through subtle reasoning or elucidated apparent contradictions. No! It was none of these things which I regard as important. Invariably, they remark with wonder on the way in which I had performed numerical calculations.

Now, I suppose that to the mathematician this particular ability is given a rating in some lower limbo of neglect. But, apparently, it is one of the things that the student can appreciate. When somebody can do mentally that which takes him considerable time and effort to carry through with paper and pencil, and when, moreover, he has not the slightest idea of what is transpiring in the teacher's brain and, therefore, finds himself face to face with the apparently marvelous, he is apt to consider his teacher a genius. Granted that this is a foible of the immature mind, does this make it any less in order to capitalize on the situation to build up one's prestige in the classroom?

Apart from the psychological advantage, the ability to calculate rapidly and efficiently is a great timesaver for the teacher of mathematics and science. There is a further advantage in that it becomes possible to check students' work with facility, at least as to order of magnitude. Finally, this ability, which seems something mysterious to the student, could lead to curiosity and a desire on his part to emulate his teacher. Such a situation provides an excellent opportunity for promoting efficiency in calculation as well as a better understanding of some basic arithmetic and algebra. With these points in mind, the following hints are offered to the teacher who wants to become a wizard at calculation in the classroom.

Limiting ourselves to operations with whole numbers between zero and one hundred, I believe the most important single thing that needs to be done is to learn the squares of all these integers. At first sight, this seems like a big order. But let us break the matter down, and it will become evident that the amount of memory work can be greatly reduced.

To start with, a person working with mathematics as his profession probably knows a number of squares already—possibly up to the square of 25. Of the remaining numbers less than a hundred, those ending in zero are no problem. For numbers ending in 5, there is the familiar rule of multiplying the ten's digit—more generally the integer represented by all the digits which precede the 5—by the next higher integer and appending 25. Thus, the square of 65 is 4225. This is easily seen by expressing the number as $10a+5$. Then,

$(10a+5)^2=100a^2+100a+25=100a(a+1)+25$.

Squaring numbers ending in 1 or 9 is readily handled by the method of squaring a binomial. For example, the square of 61 would be the same as the $(60+1)^2=3600+120+1$. In other words, we square 60, add twice sixty and then add 1, thus obtaining 3721. For 59, the result would be found by squaring $60-1$. Hence you subtract the 120 from 3600 instead of adding it, then add 1, getting the answer of 3481. The same procedure can be followed with many other numbers as well.

Neglecting such an extension, we are left with the numbers ending in 2, 3, 4, 6, 7, and 8. However, the squares of numbers from 40 to 60 are given by a very simple rule and likewise those from 90 to 100. For numbers from 40 to 60 we use 50 as a reference point. Thus, to find the square of 43, we note that this is 7 from 50, whose square is 2500. To calculate the square of 43, take 7 from 25 and append the square of 7; the answer is 1849. If the figure to be squared had been 57, the answer would have been obtained by adding 7 to 25 and appending 49; the result is 3249. The basis of this rule can be seen from the formula,

$$(50\pm a)^2=2500\pm 100a+a^2=100(25\pm a)+a^2.$$

For 90 to 100, the procedure is similar except that we go by steps of 200 instead of 100. Thus, the square of 94 is found by taking 12 from 100 and appending 36; the answer is 8836. The formula that justifies this rule is $(100-a)^2=10{,}000-200a+a^2=100\,(100-2a)+a^2$.

There are other helpful devices we can call to our aid. It may be noted that all the endings of squares vary in sequence in a period of 25. Thus, on either side of any multiple of 25, the squares of any two numbers a corresponding distance from that multiple of 25 have the same two digits in the last two places. Examples are:

$24^2=576$ and $26^2=676$
$48^2=2304$ and $52^2=2704$
$68^2=4624$ and $82^2=6724$.

As an illustration of how such information can be helpful, suppose we have momentarily forgotten the square of 66. We note that this figure is $50+16$ and that, therefore, it must have the same last two digits in its square as are found in 16 squared. So it must end in 56. Possibly this hint recalls the value 4356.

With the squares more or less firmly in mind, we are ready to do mental calculations by the method of the sum and difference of two numbers. Given a problem, 56×62, where the interval between the numbers is even, we know at once that the product is 9 less than the square of 59; $56\times62=(59-3)(59+3)=3481-9=3472$. If the interval is odd—for example, if we are trying to multiply 47 by 52—we can first find 48 times 52 and then subtract 52 because $47\times52=(48-1)\times52$; 48 times 52 is $2500-4$; 52 less is 2444.

Using the rule of squaring a number ending in 5, it is possible to do calculations which to the uninitiated seem remarkable. For example, to find the square of 545, we need only remember that 54 squared is 2916, add another 54 to get 2970 and append 25 to arrive at the final answer of 297,025.

The squares are likewise very valuable in the inverse operation of extracting square roots. For instance, given a right triangle with legs 4 and 5, we know that the hypotenuse is the square root of 41. But since 64 squared is 4096 and this is very close to 4100, the answer is approximately 6.4. In general, with a little practice, it is not difficult to obtain mentally the square root of a number simply by having in mind the squares of the numbers from one to one hundred. In any event, even should there be some need to write down a few numbers, the process is reduced to a brief set of calculations.

Familiarity with squares is useful in working with powers of the smaller numbers, such as 2, 3, 4, etc. For example, 2 to the tenth power is immediately seen to be 1024, since this is the square of 32 which is 2 to the fifth power. Actually, it

is not difficult to calculate 2 to the twentieth power mentally, treating 1024 as 1000+24 and squaring as a binomial, because it is easy to add 1,000,000, 48,000, and 576. The result is 1,048,576.

As another example of what can be done along this line, consider the problem of obtaining 3 to the tenth power. Since 3^4 is 81 and 81 squared is 6561, 3^{10} is 9 times this amount. Therefore, keeping this figure in mind and multiplying by 9, we can obtain the answer 59,049.

The squares, then, play an important role in these mental calculations. However, there are other methods that can be used to advantage as well. One of the simplest is multiplication by 25. When a teacher has the problem

$$25 \times 84.3621$$

and proceeds to write the answer directly as 2109.0525, students may be amazed. Actually, all he has done is divide by 4 and shift the decimal point two places.

To multiply by a number such as 99, we use in effect the distributive law, replacing 99 by 100−1. In other words, we add two zeros to the number and subtract the number. Thus, $99 \times 84 = (100-1) \times 84 = 8400 - 84 = 8316$. Similarly, to multiply 79×63, we find the result of 80 times 63 and subtract 63.

It is good to know the common square roots, such as $\sqrt{2} = 1.4142$ and $\sqrt{3} = 1.7321$. In this way, it is often possible to write down answers directly without having to go through a long-drawn-out process.

These few hints do not represent the end of the line by any means as far as facility with figures is concerned. But they do provide a good starting point. It is possible with practice to build up a creditable ability in mental calculation; and, as mentioned in the earlier part of this paper, such an ability, while lowly in appearance, can and does have a salutary effect in the classroom.

Illustrating locus problems

by Frances Steffen, Mason City High School, Mason City, Iowa

Many students in my plane geometry classes have had difficulty in visualizing compound locus problems and drawing illustrations for them. Even when they do grasp the idea, it is time consuming to illustrate all possible cases.

Last year I had the students draw the following four simple, basic locus figures in ink on half-sheets of onionskin paper:

1. The locus of points one inch from a given point.
2. The locus of points one-half inch from a given line.
3. The locus of points equidistant from two parallel lines.
4. The locus of points equidistant from two given points.

By placing one figure on another, it was comparatively easy to see what points would satisfy both given conditions. The students carried these illustrations in their notebooks for repeated use.

Once these figures are constructed, they are instantly available at all times and can be used in many ways. They can be combined with each other or with another drawing. They can be moved about to show different relative distances and special cases. They can be used instead of a drawing for a particular problem, or they can be used to facilitate the drawing of the problem.

We found these devices a great time saver and an aid in clarifying problems.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

JOINT MEETING WITH NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTY-FIRST SUMMER MEETING

August 21–23, 1961
University of Toronto, Toronto, Canada
Father John C. Egsgard, C.S.B., St. Michael's College School, 1515 Bathurst Street, Toronto 10, Canada

FORTIETH ANNUAL MEETING

April 15–18, 1962
Jack Tar Hotel, San Francisco, California
Kenneth C. Skeen, 3355 Cowell Road, Concord, California

Other professional dates

Illinois Council of Teachers of Mathematics

April 22, 1961 Illinois State Normal University, Normal, Illinois
April 29, 1961 Sterling Township High School, Sterling, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

The Greater Cleveland Council of Teachers of Mathematics

April 18, 1961
Bedford High School, Bedford, Ohio
Bessie Kisner, Strongsville High School, Strongsville, Ohio

Men's Mathematics Club of Chicago and Metropolitan Area

April 21, 1961 and
May 19, 1961
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

California Mathematics Council

April 28–30, 1961
Santa Monica, California
Miss Mary Lou Hood, 3414 North Eckart, South San Gabriel, California

Pennsylvania Council of Teachers of Mathematics

April 29, 1961
Clarion State College, Clarion, Pennsylvania
Earl F. Myers, 322 13th Avenue, New Brighton, Pennsylvania

Association of Mathematics Teachers of New York State

May 5–6, 1961
Hotel Syracuse, Syracuse, New York
Edward E. Sherley, Mt. Pleasant High School, Schenectady 3, New York

Michigan Council of Teachers of Mathematics

May 5–7, 1961
M. E. S. Camp, Saint Mary's Lake, Battle Creek, Michigan
Mrs. Marjorie Pickering, 5198 Coldspring, Birmingham, Michigan

Chicago Elementary Teachers Mathematics Club

May 8, 1961
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Mildred C. Rogers, Warren Elementary School, 9210 South Chappel Avenue, Chicago 17, Illinois

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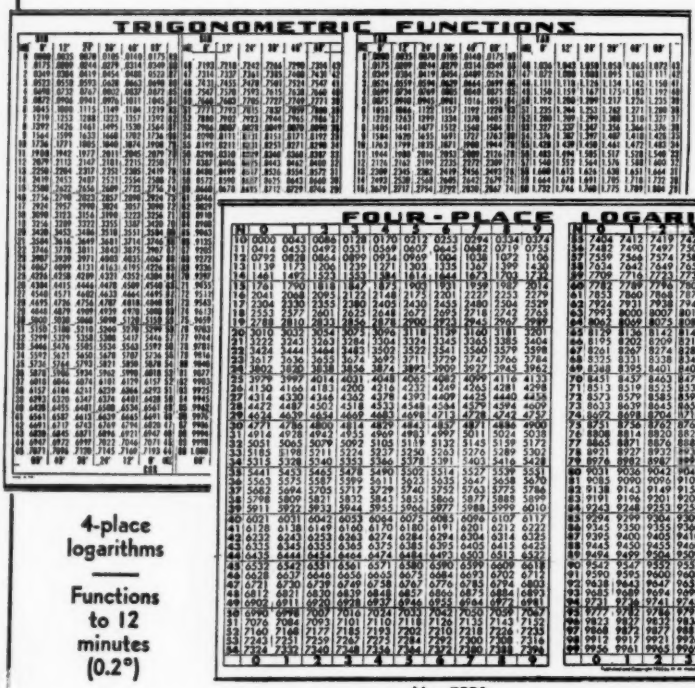

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